

Beam-beam studies for the High-Luminosity and High-Energy LHC, plus related issues for KEKB

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CERN Accelerator Physics Forum

Thanks to F. Zimmermann, R. Tomas, R. Calaga, O. Dominguez, O. Bruening and members of ABP group

HE-LHC

- $E=16.5\text{TeV}$
- Radiation damping time is 1h in the transverse.

Updated parameter list for LHC energy upgrade at 33 TeV centre-of-mass energy

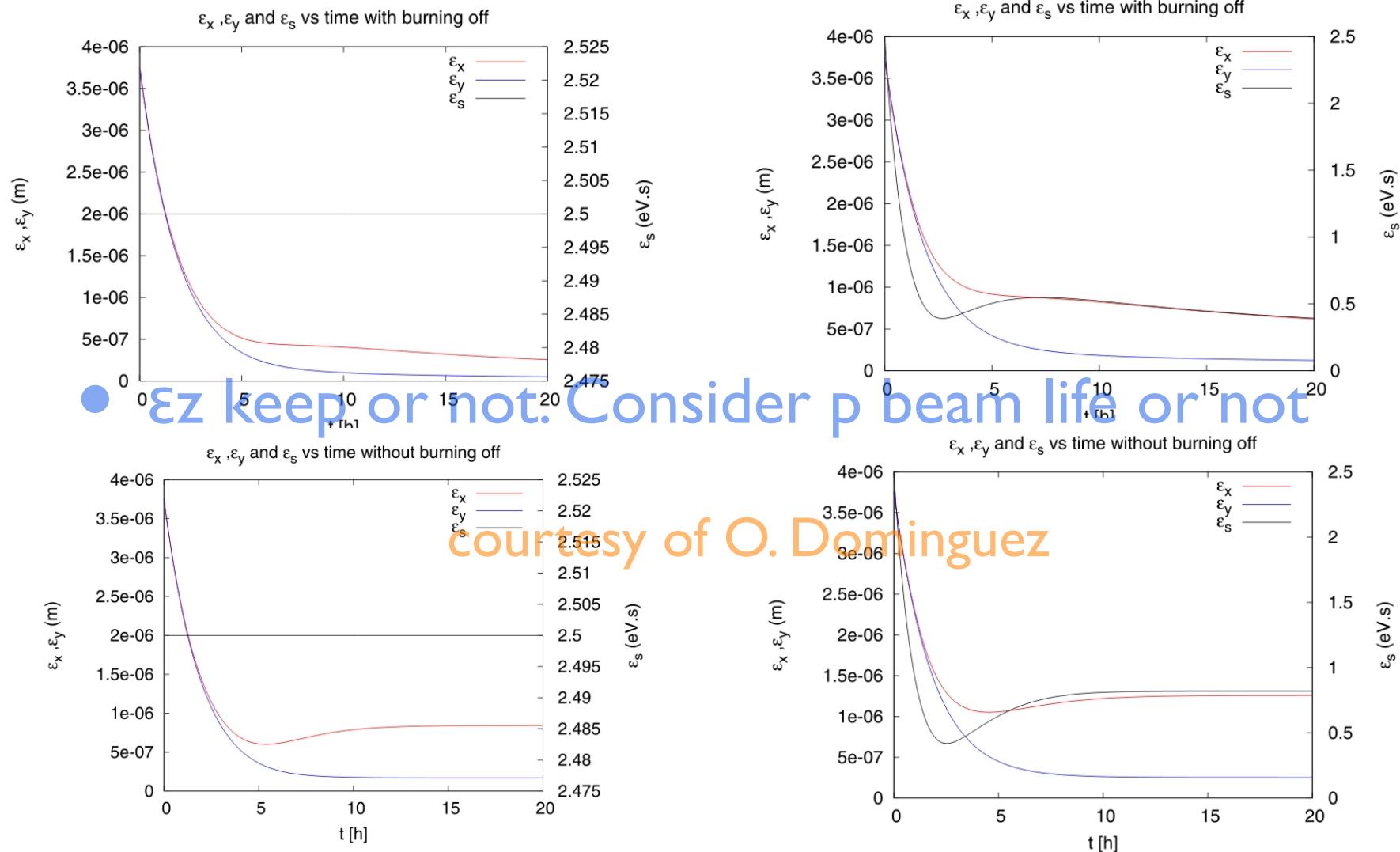
	nominal LHC	LHC energy upgrade
beam energy [TeV]	7	16.5
dipole field [T]	8.33	20
dipole coil aperture [mm]	56	40
beam half aperture [cm]	2.2 (x), 1.8 (y)	1.3
#bunches	2808	1404
bunch population [10^{11}]	1.15	1.29
initial transverse normalized emittance [μm]	3.75	3.75, 1.84
initial longitudinal emittance [eVs]	2.5	4.0
number of IPs contributing to tune shift	3	2
initial total beam-beam tune shift	0.01	0.01 (x & y)
maximum total beam-beam tune shift	0.01	0.01
RF voltage [MV]	16	32
rms bunch length [cm]	7.55	6.5
rms momentum spread [10^{-4}]	1.13	0.9
IP beta function [m]	0.55	1 (x), 0.43 (y)
initial rms IP spot size [μm]	16.7	14.6 (x), 6.3 (y)
full crossing angle [μrad]	285 (9.5 σ_{xy})	175 (12 σ_{x0})
Piwinski angle	0.65	0.39
geometric luminosity loss from crossing	0.84	0.93
stored beam energy [MJ]	362	478.5
SR power per ring [kW]	3.6	62.3
dipole SR heat load dW/ds [W/m/aperture]	0.21	3.64
energy loss per turn [keV]	6.7	207.1
critical photon energy	44	576
longitudinal SR emittance damping time [h]	12.9	0.98
horizontal SR emittance damping time [h]	25.8	1.97
initial longitudinal IBS emittance rise time [h]	61	64
initial horizontal IBS emittance rise time [h]	80	80
initial vertical IBS emittance rise time [h] ($\kappa=0.2$)	~400	~400
<i>note: IBS rise times > SR damping times</i>		
events per crossing	19	76
initial luminosity [$10^{34} \text{ cm}^{-2}\text{s}^{-1}$]	1.0	2.0
peak luminosity [$10^{34} \text{ cm}^{-2}\text{s}^{-1}$]	1.0	2.0
beam lifetime [h]	46	12.6
integrated luminosity over 10 h [fb^{-1}]	0.3	0.5

Random Excitation

- Synchrotron radiation was a dominant excitation source in traditional electron rings.
- Intra-beam scattering is dominant in recent electron storage rings especially for the vertical emittance. It is also dominant in ion beam with a high Atomic number (RHIC).
- Radiation excitation is weaker than that of intra beam in proton beam of HE-LHC.
- The excitation of the intra-beam scattering depend on the beam emittance.

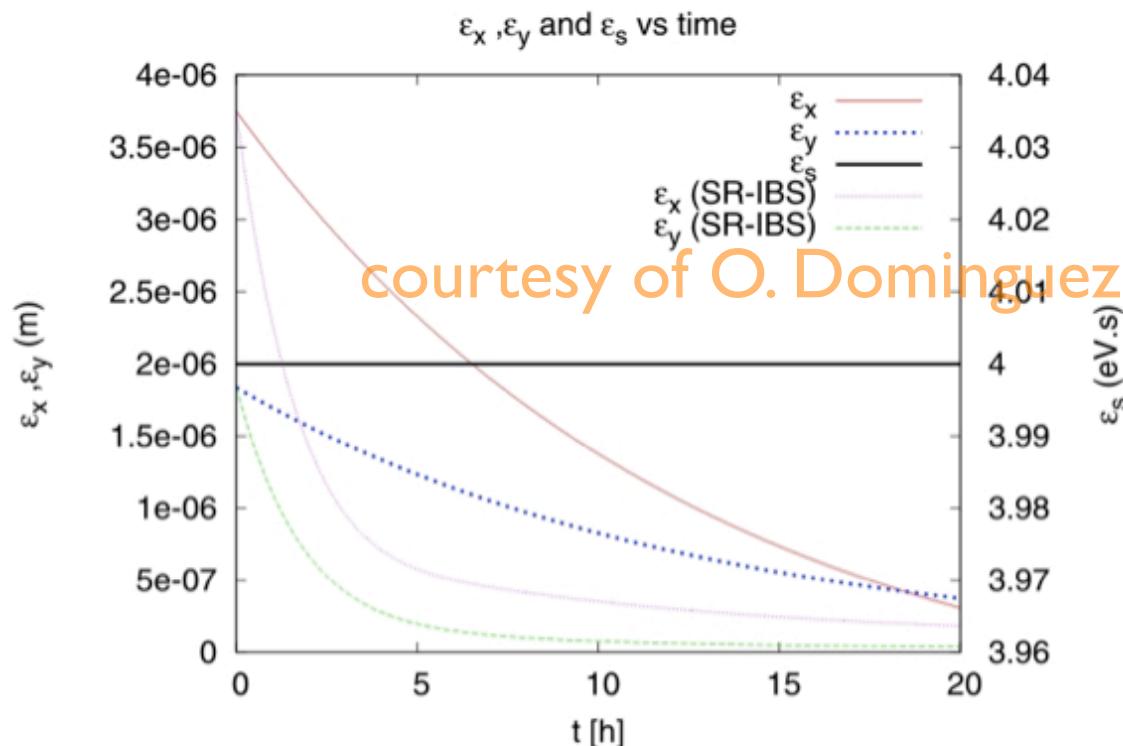
Beam size evolution with radiation damping and IBS

O. Dominguez and F. Zimmermann



Control transverse excitation

- Emittance is controllable by applying external fluctuation (kicker).



Beam-beam simulation for HE-LHC

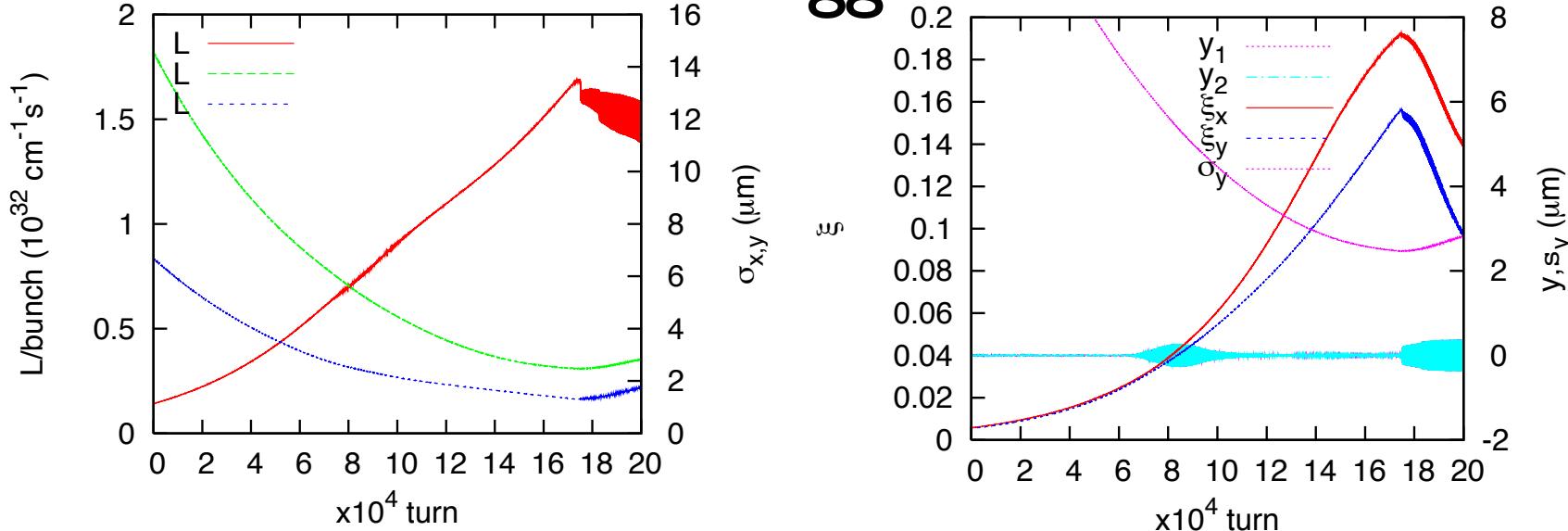
- Strong-strong simulation with a code BBSS. Single IP.
- The excitation rate are not put properly perhaps. I will calculate with more possibility in the future.
- Anyway the damping times must be faster in the present simulation to have results. Which is important excitation rate or ratio of excitation and damping?

Outlook of the simulation result

- Dipole oscillation arises at very high beam-beam parameter >0.1.
- Small dipole oscillation arises around $\xi \sim 0.03$ for excitation ON, but disappear.
- IBS limits the beam-beam parameter, maybe geometrically.

Assume 200 times faster damping time

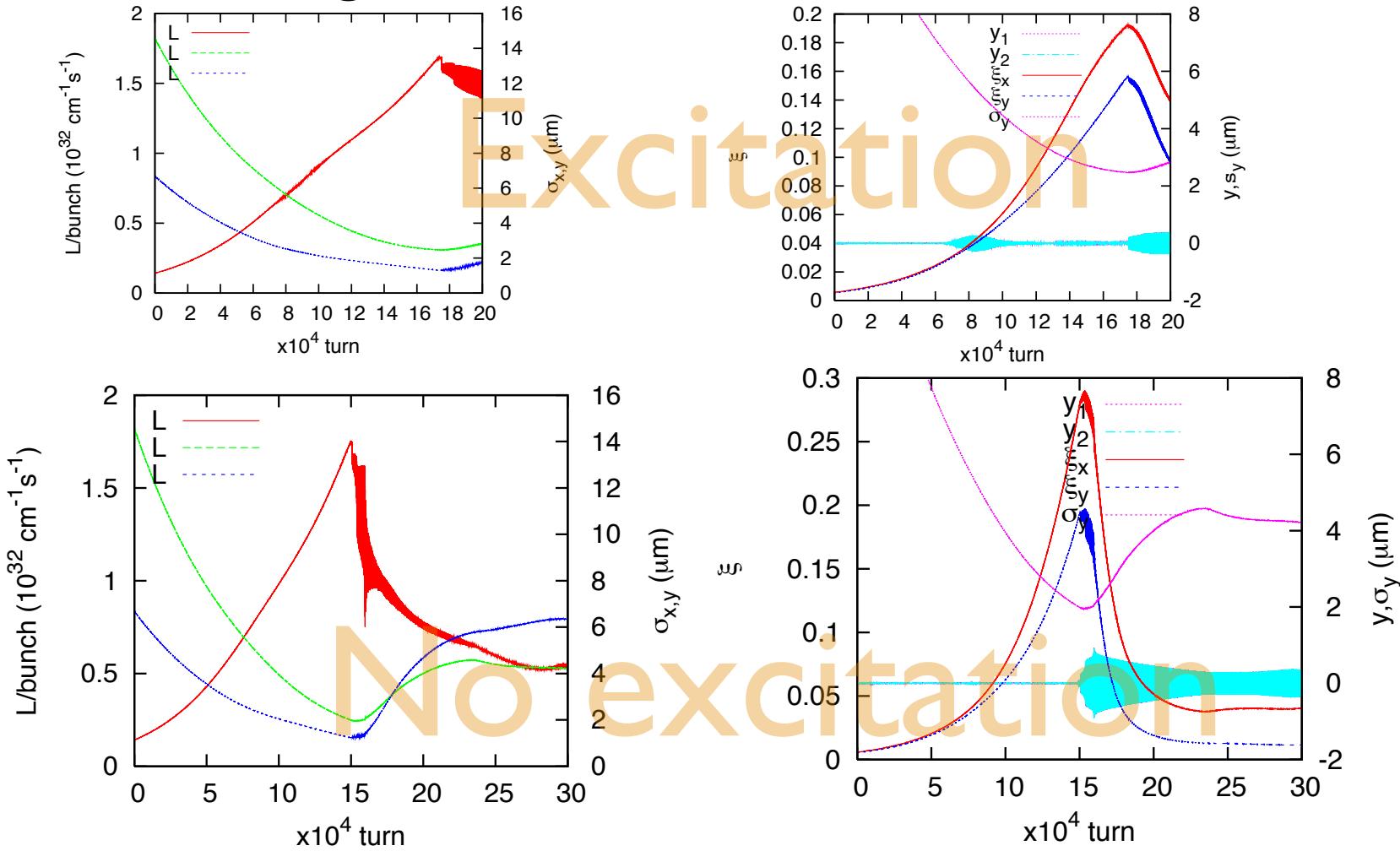
20 times bigger excitation



- Dipole oscillation limit the luminosity. The beam-beam parameter is very high $\xi > 0.1$.
- The dipole oscillation was seen at $\xi > 0.05$ in a flat beam such as lepton colliders for $\sigma_z \ll \beta_y$.

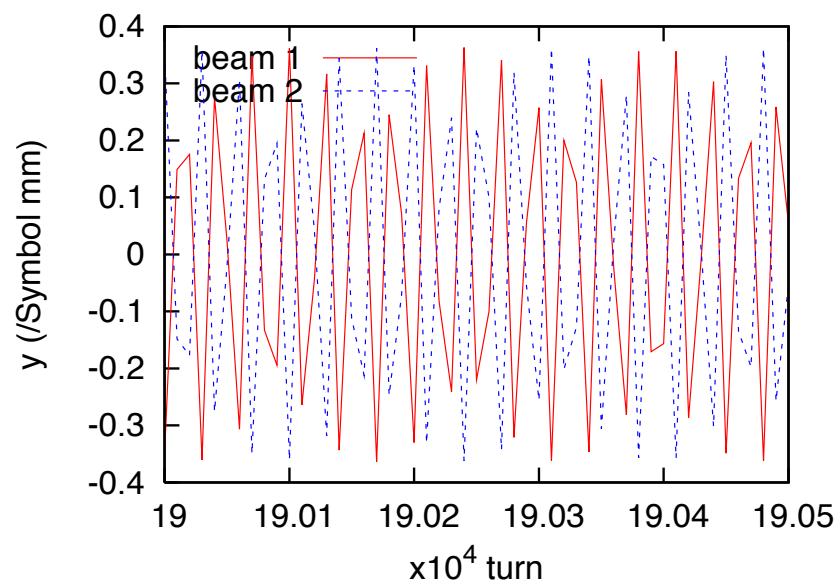
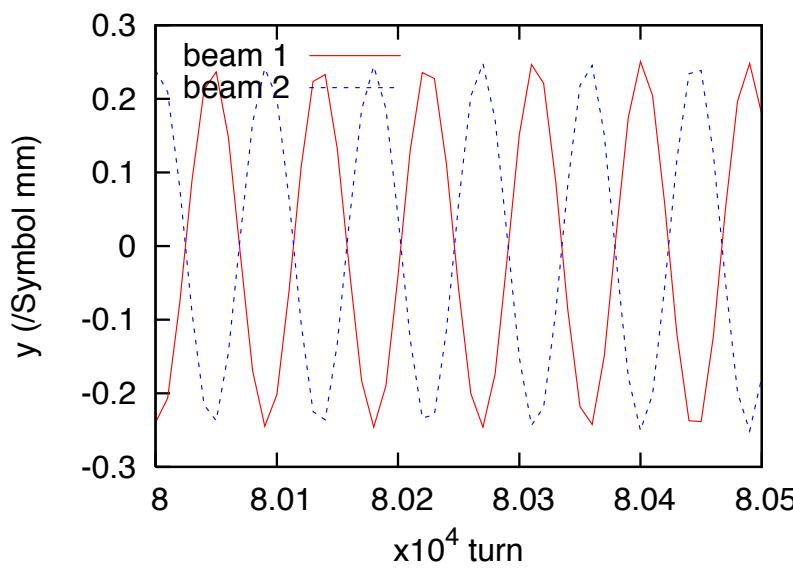
Excitation ON/OFF

- No big difference.



Dipole oscillation

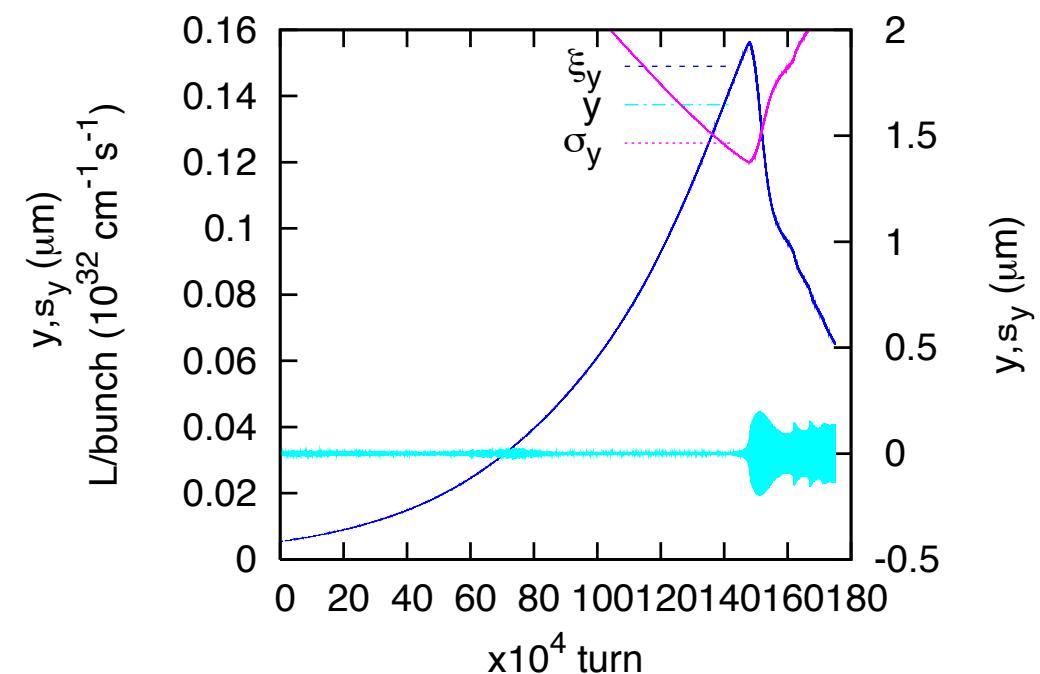
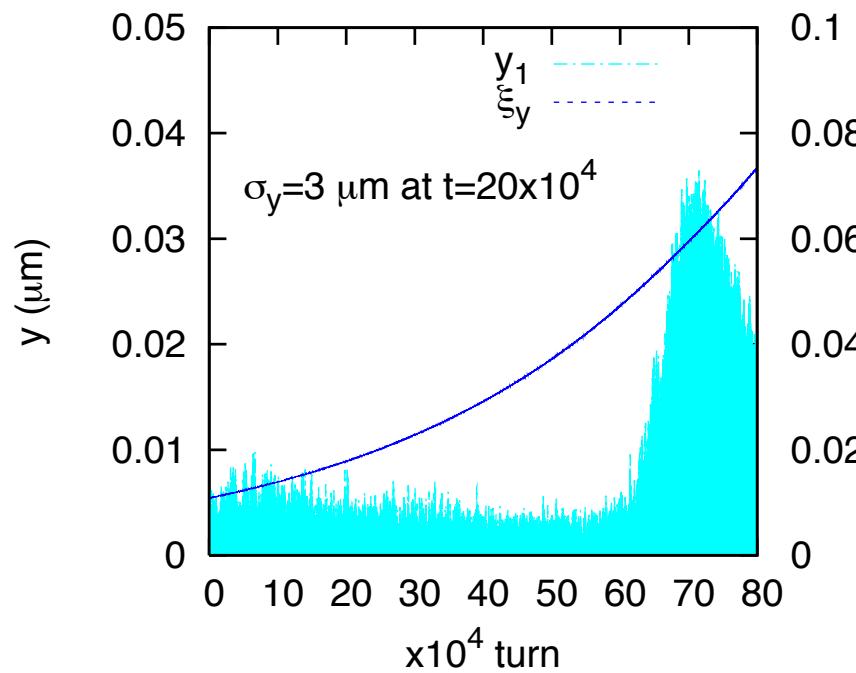
- π mode frequency seems to shift.



20 times faster damping

2 times bigger excitation

- Excitation ON/OFF
- No remarkable difference



Tentative result for HE-LHC

- Coherent instability arises at $\xi \sim 0.15$.
- IBS equilibrium emittance is $\sim 1/10$ of ϵ_0 .
- Incoherent growth time is 1 day for $\xi = 0.03$ as shown later.
- The beam-beam effect is weak for an ideal case treated here. Luminosity is determined by IBS equilibrium emittance geometrically, if $\xi < 0.03$.
- I would like to a systematic study for diffusion rate and damping rate in the beam-beam environment.

x-y coupling at IP

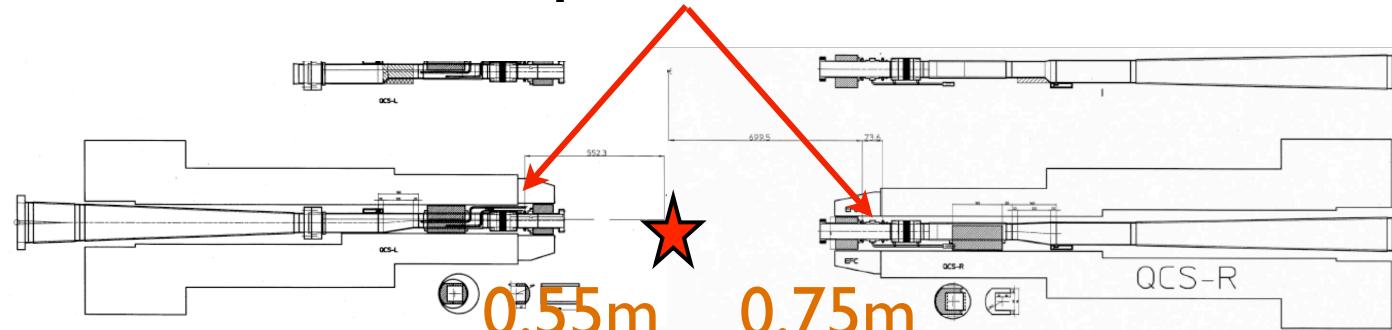
- x-y coupling affects the beam-beam performance essentially in KEKB. It is effect of beam-beam dynamics, but not that of geometric.
- Parametrization of x-y coupling

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = RB \begin{pmatrix} X \\ X' \\ Y \\ Y' \end{pmatrix} \quad R = \begin{pmatrix} r_0 & 0 & r_4 & -r_2 \\ 0 & r_0 & -r_3 & r_1 \\ -r_1 & -r_2 & r_0 & 0 \\ -r_3 & -r_4 & 0 & r_0 \end{pmatrix}$$

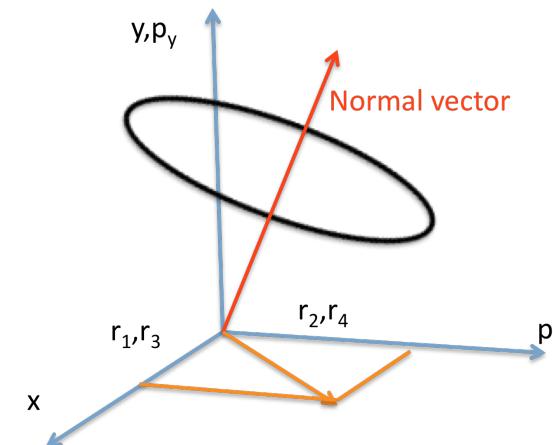
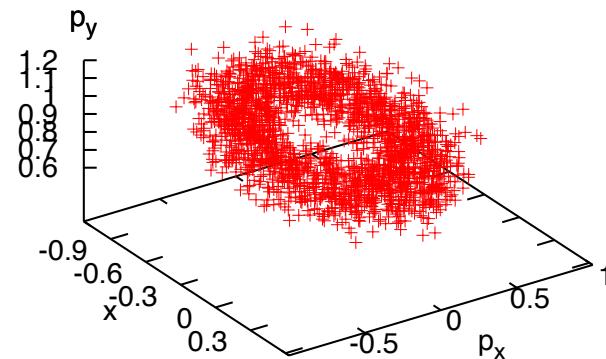
$$B = \begin{pmatrix} \sqrt{\beta_x} & 0 & 0 & 0 \\ -\alpha_x/\sqrt{\beta_x} & 1/\sqrt{\beta_x} & 0 & 0 \\ 0 & 0 & \sqrt{\beta_y} & 0 \\ 0 & 0 & -\alpha_y/\sqrt{\beta_y} & 1/\sqrt{\beta_y} \end{pmatrix}$$

Measurement of IP coupling at KEKB

- Use the Octopos monitor both side of IP.



- Reconstruct 4 dimensional phase space.



R's parameters correspond to the normal vector in $x-p_x-y$ ($r1, r2$) and $x-p_x-p_y$ ($r3, r4$) phase space.

Correlation matrix

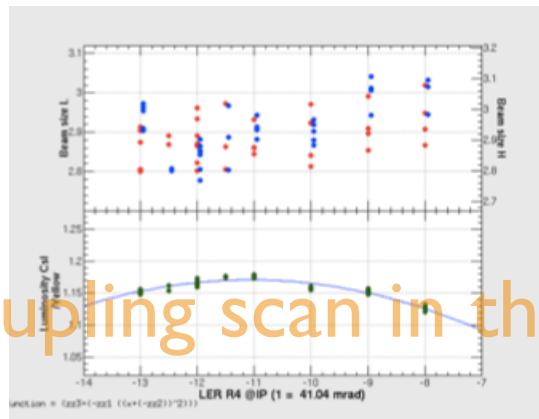
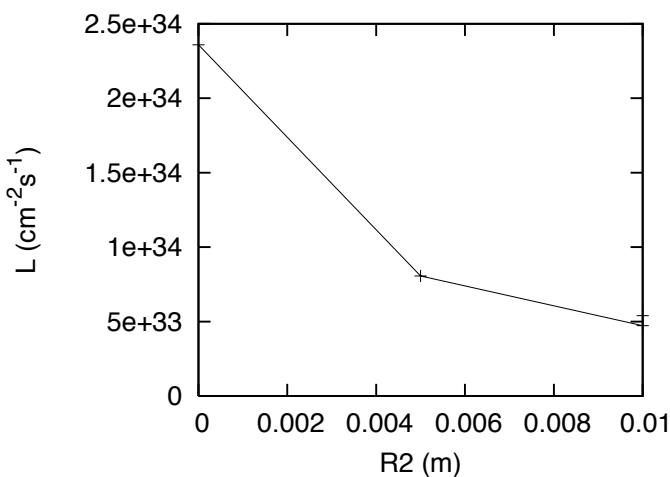
$$\begin{pmatrix} \langle xx \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle & \langle x'y \rangle & \langle x'y' \rangle \\ \langle xy \rangle & \langle x'y \rangle & \langle yy \rangle & \langle yy' \rangle \\ \langle xy' \rangle & \langle x'y' \rangle & \langle yy' \rangle & \langle y'y' \rangle \end{pmatrix} = RB \begin{pmatrix} \langle X^2 \rangle & 0 & 0 & 0 \\ 0 & \langle X^2 \rangle & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} B^t R^t$$

Excite only X mode

- Correlation matrix for the measured phase space variables give Twiss parameters including R's.
- Chromatic coupling at IP was measured and corrected by skew sextupole magnets. Luminosity increased in KEKB.

Simulations of x-y coupling effects in beam-beam interaction

- Day by day tuning in KEKB is spent for the coupling scan.



Coupling scan in the operation

chromatic coupling scan

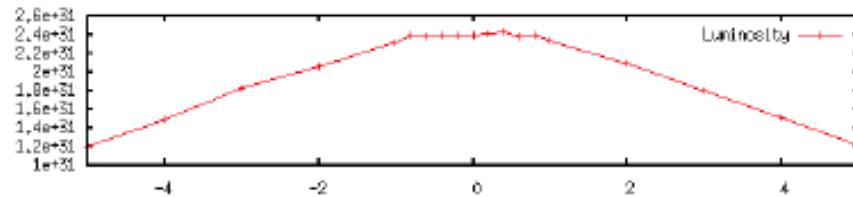


Fig.6 Horizontal beam size, Vertical beam size and Luminosity vs. $dR_2/d\delta$

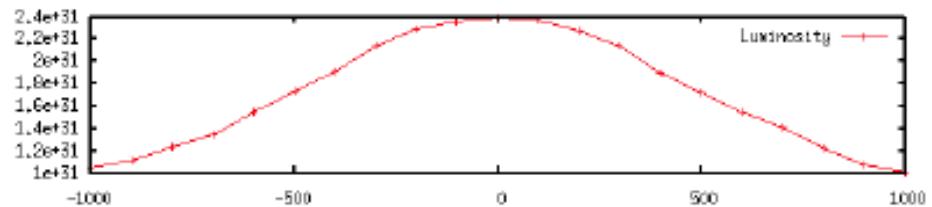


Fig.8 Horizontal beam size, Vertical beam size and Luminosity vs. $dR_4/d\delta$

x-y coupling in HE-LHC

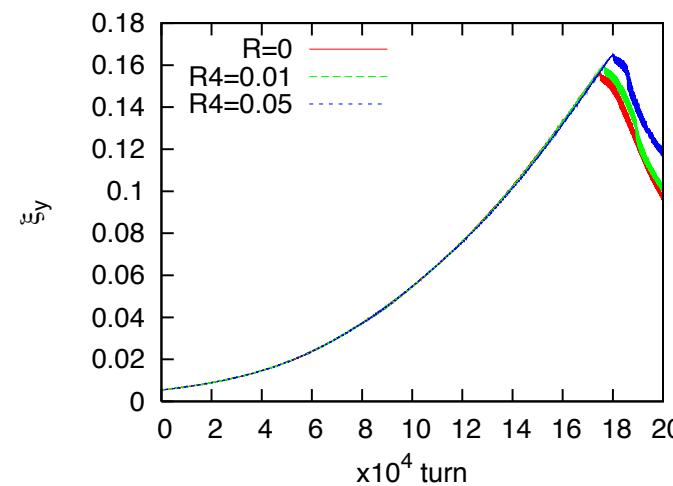
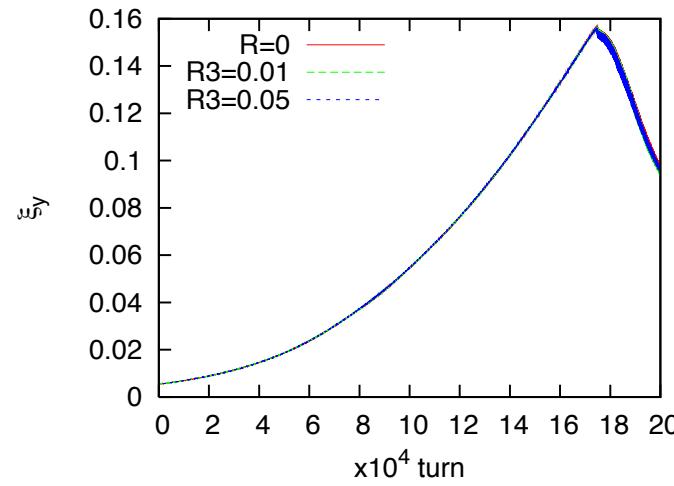
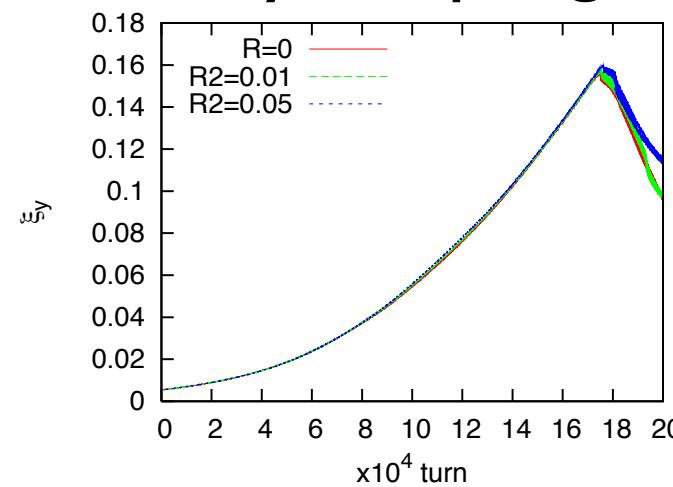
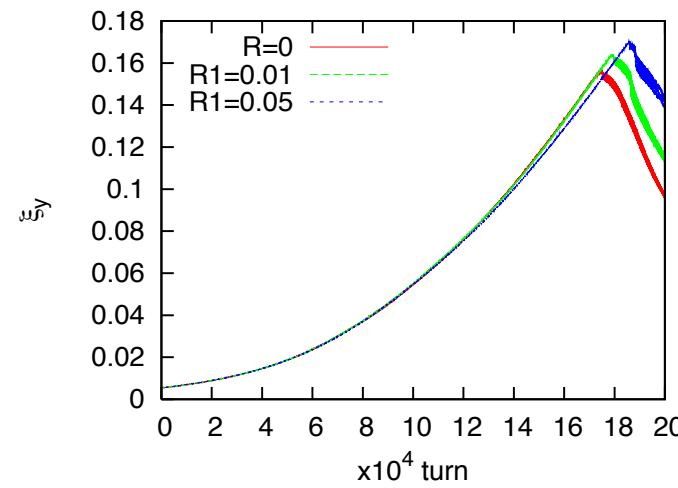
- How x-y coupling affects LHC performance?
- Model with 200 time faster damping.

Outlook of the results

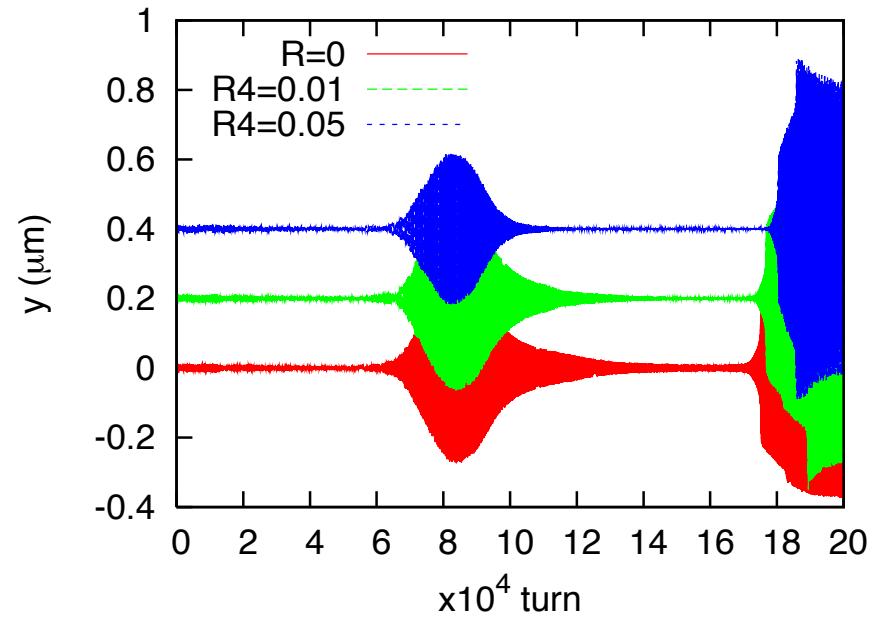
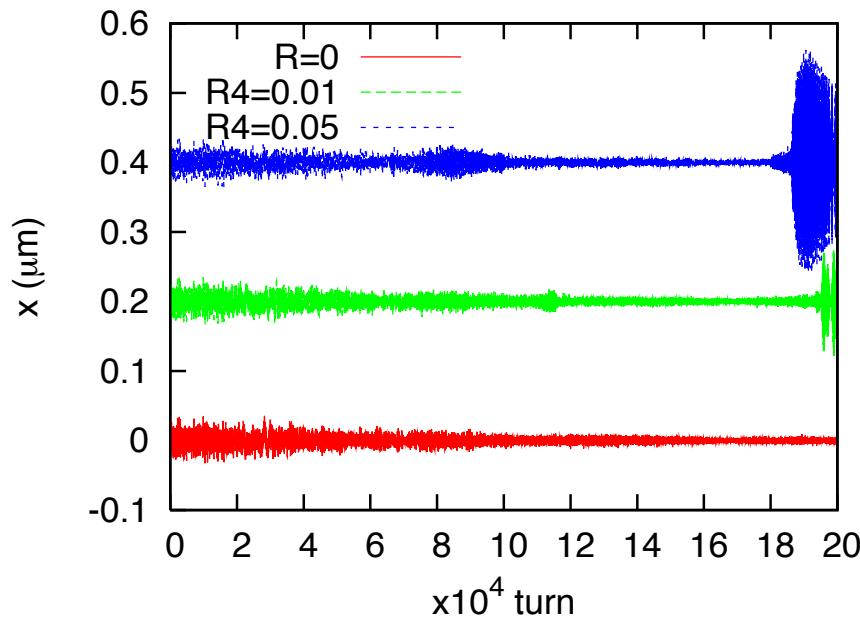
- The answer was very weak.
- The reason may be in the round beam.

x-y coupling at IP

- No remarkable effect for x-y coupling



Dipole oscillation



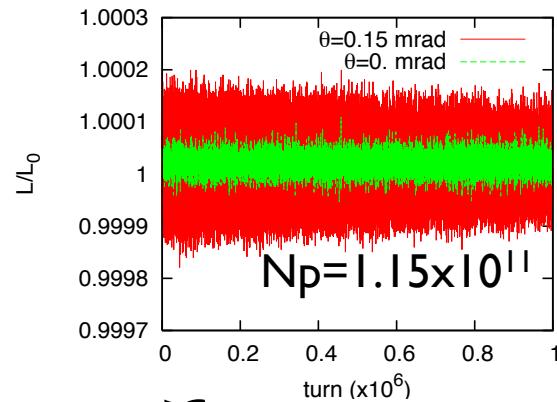
- Similar behavior in all R's.
- Horizontal coherent motion is seen for coupled cases.

HL-LHC

Studies in 2008, luminosity by simulation

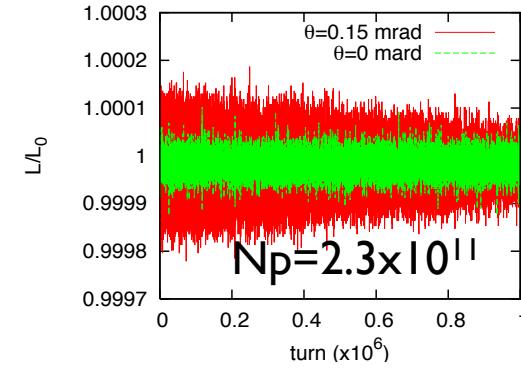
- Nominal $1 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$
- Crab cavity(cc) $1.15 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$
- UT $\beta=0.5\text{m}$ $N_p=1.7 \times 10^{11}$, $\theta=315 \mu\text{rad}$,
 $L=2.08 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$, $2.5 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$ (cc)
- ES cc $1.2 \times 10^{35} \text{ cm}^{-2}\text{s}^{-1}$
- LPA1 $0.58 \times 10^{35} \text{ cm}^{-2}\text{s}^{-1}$
- LPA2 $0.71 \times 10^{35} \text{ cm}^{-2}\text{s}^{-1}$

Beam-beam limit in the nominal LHC simulations $\xi \sim 0.03$



$\xi = 0.004$

0.024

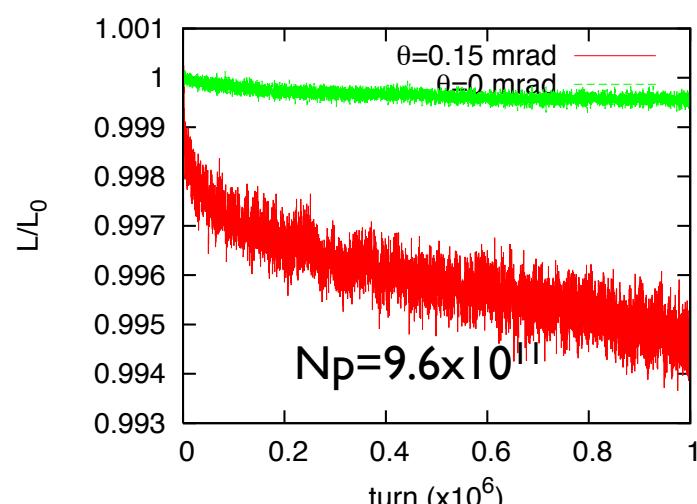
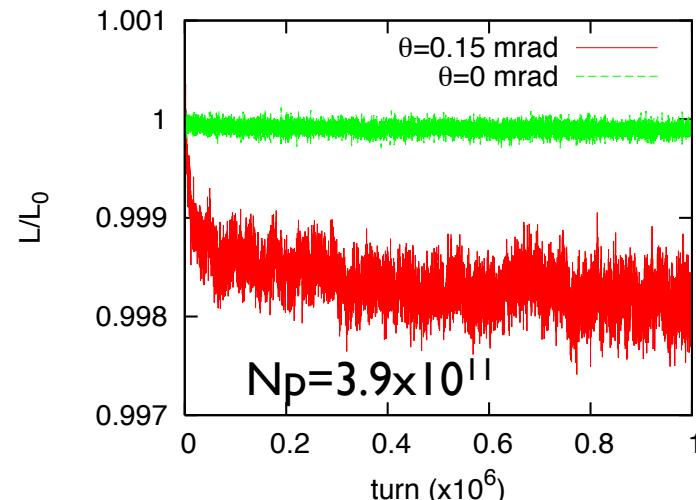
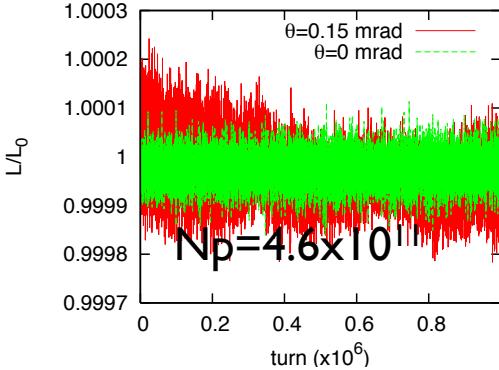


0.008

0.016

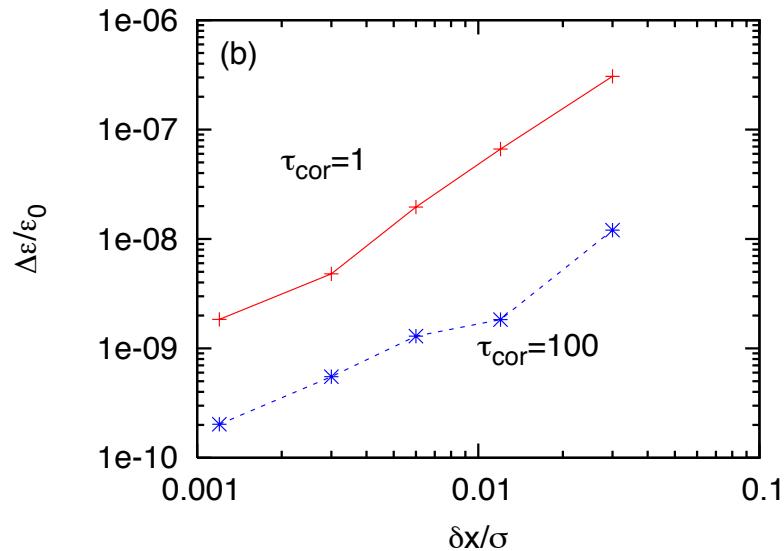
0.032

EPAC08



Crab cavity noise in LHC

- Collision offset fluctuates turn by turn.
Emittance growth.



PAC'07

- Tolerance is $\Delta x/\sigma x = 0.1\%$ for turn by turn noise.

Crab cavity noise studies in KEKB

R.Tomas et al.

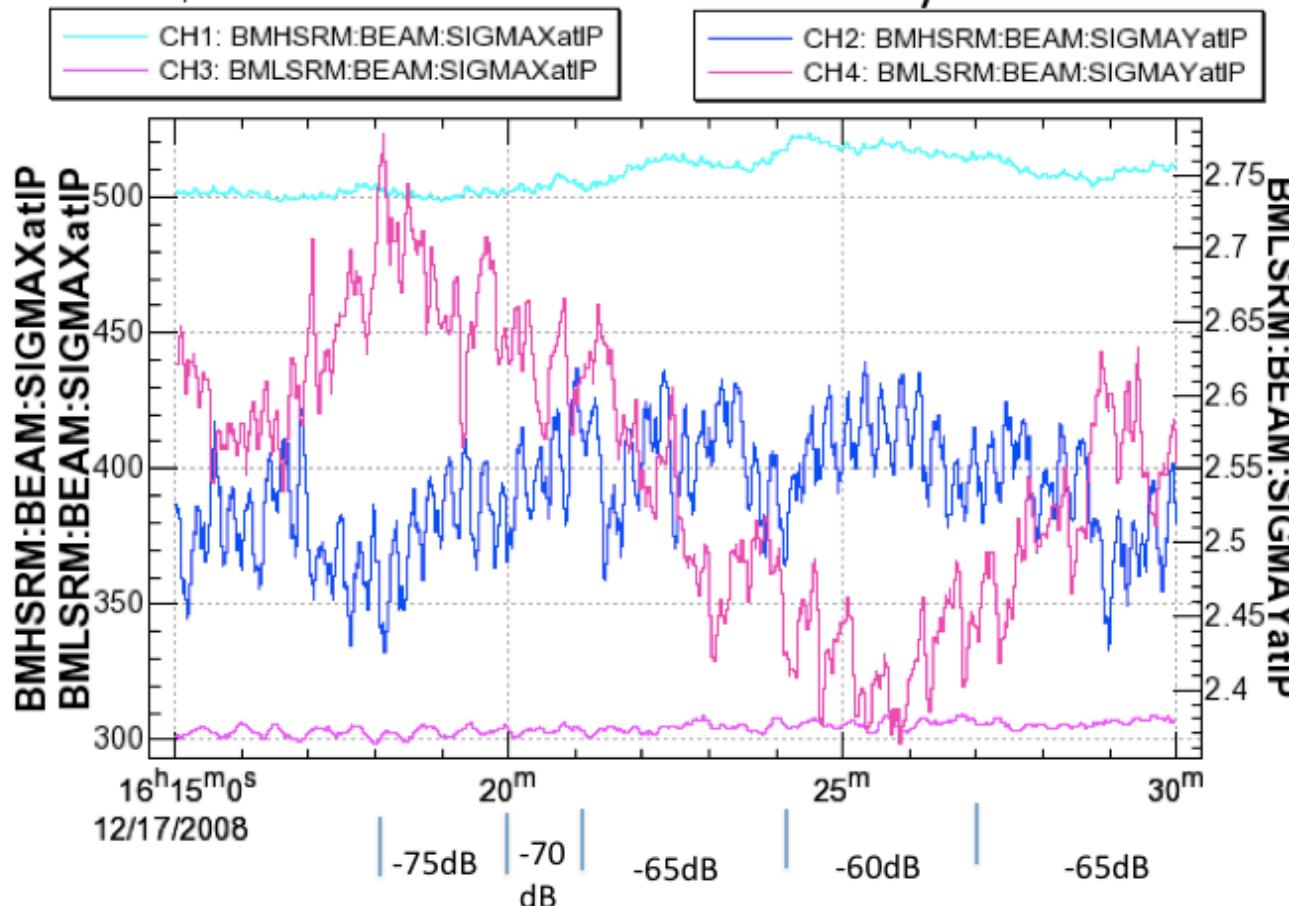
HER crab controlled phase noise

HER: $v_x: .5113, v_y: .6062$
LER: $v_x: .5056, v_y: .5833$

39.1kHz (HER: v_y)

2008/12/17

1600/1037mA (L/H)
1585 bunches



Measurement and simulation for the crab phase noise

Measurement

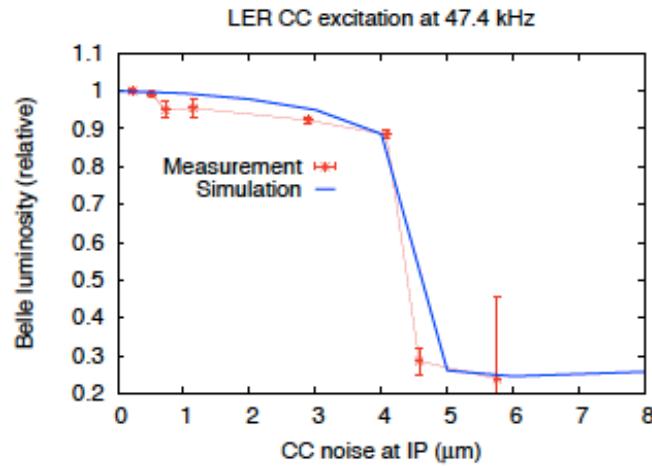


FIG. 2: Luminosity versus LER crab cavity noise as extrapolated to IP displacement. The noise frequency is close to the LER horizontal tune.

Simulation

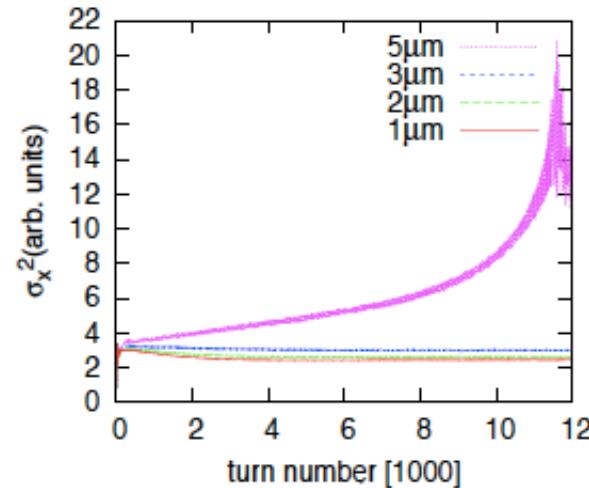
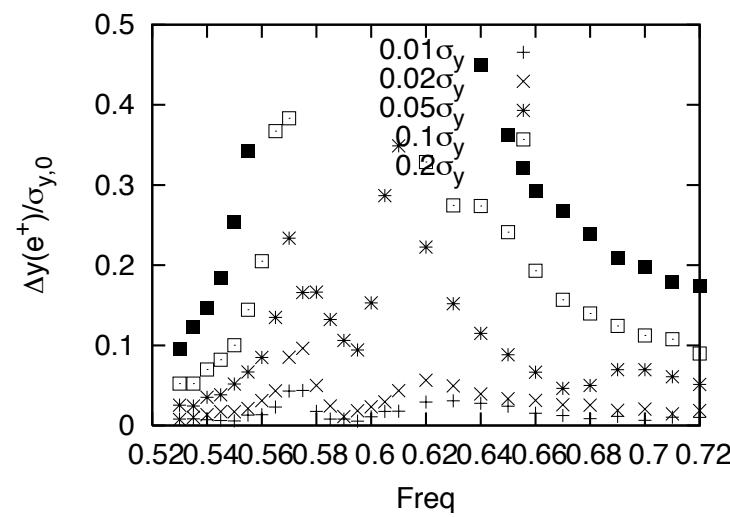
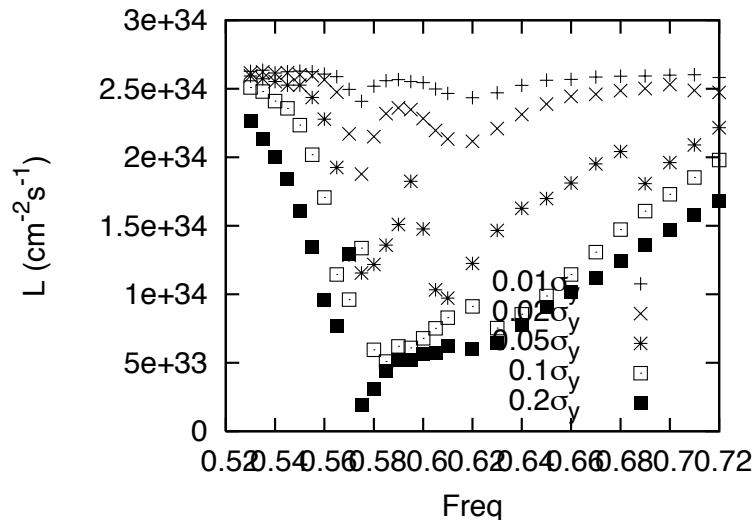


FIG. 3: Simulated beam size versus turn number for different noise amplitudes, clearly showing the existence of a threshold for the onset of the instability.

The details were given by R.Tomas.

Feed back noise studies

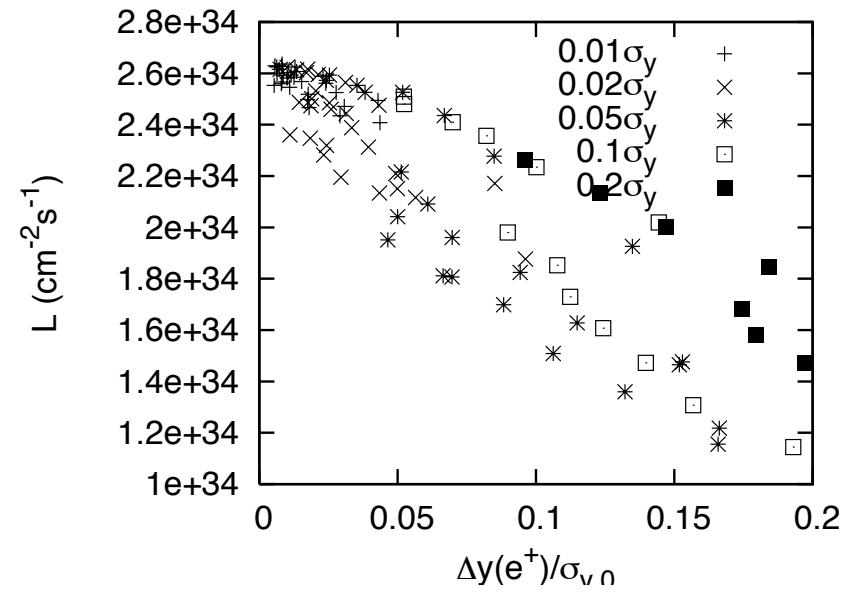
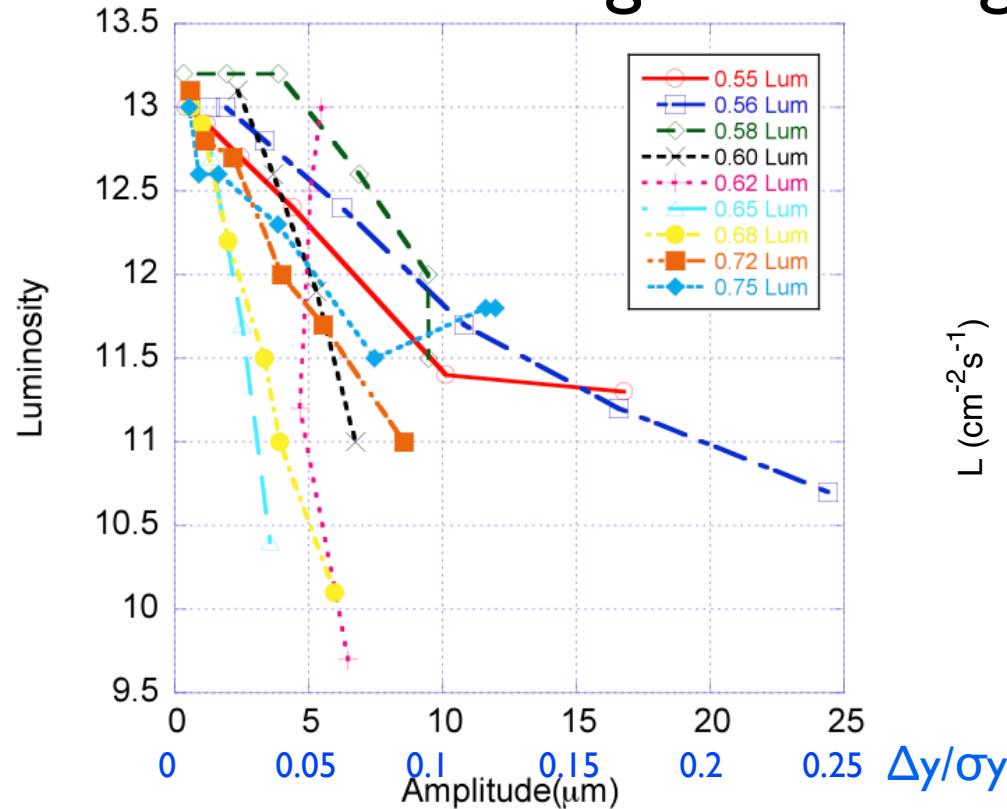
- We doubted that a fast noise degrade KEKB performance.
- Vertical beam size is small so that vertical noise is sensitive for the luminosity degradation.
- Sinusoidal noise is considered first.



Luminosity degradation in measurements and simulation

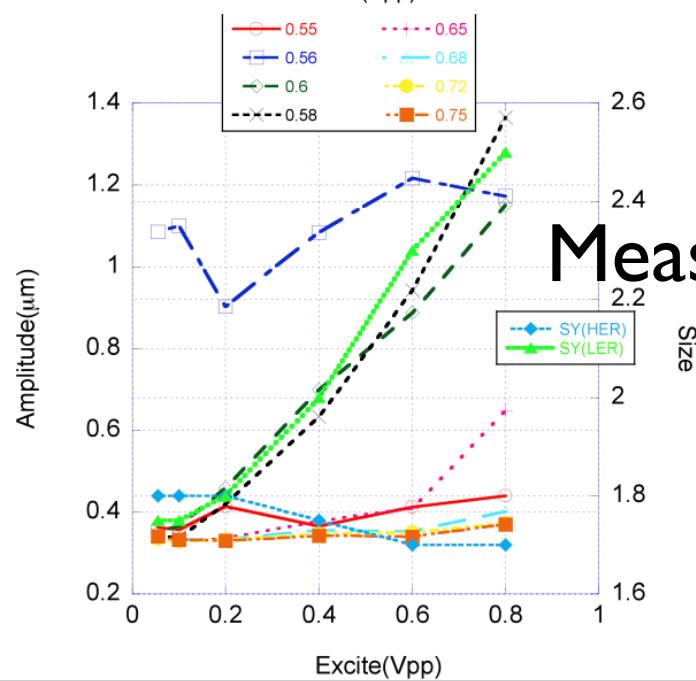
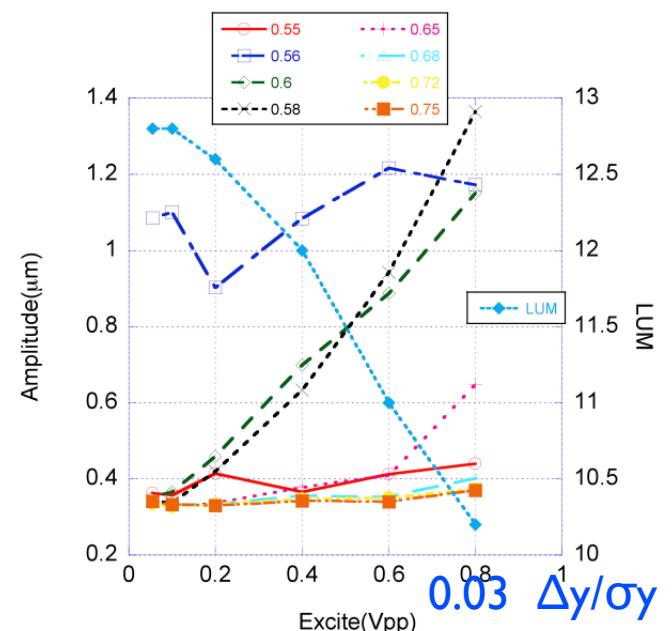
by Tobiyama and Ohmi in 24 June 2010
The final week of the operation.

Relative degradation agrees

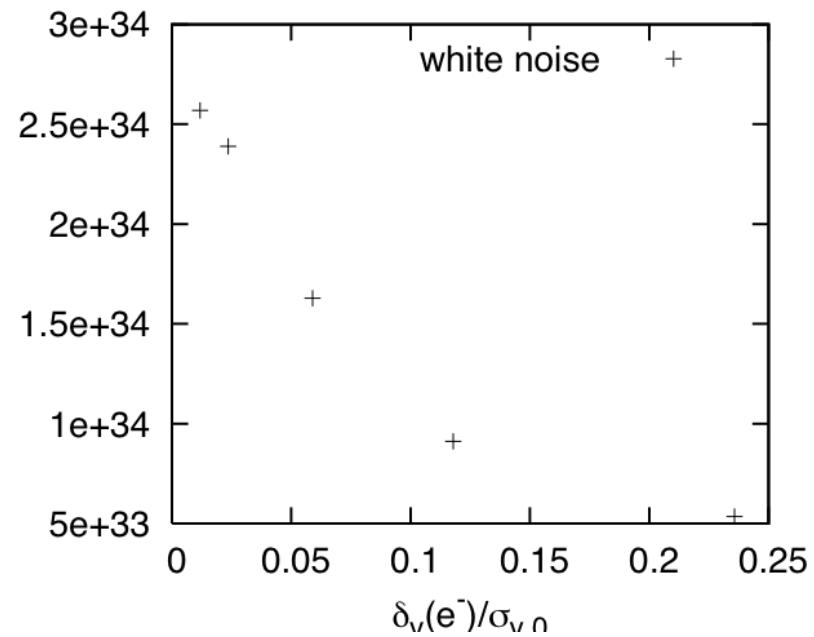


Machine condition was not the best.
Lower current due to electricity limit.

White noise



Measurement



Simulation

What degrade the luminosity in KEKB

- KEKB achieved the luminosity $2.1 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$, twice of the design. The crab cavity contributes the luminosity.
- The luminosity did not increase $> 3 \times 10^{34} \text{ cm}^{-2}\text{s}^{-1}$ as is expected.
- Tobiyama's comment. At least the feedback system does not give such strong noise into the beam.
- Other sources?
- Coupling is still unclear, but is studied much.

Crab waist

- Condition: Large Piwinski angle, **low beta smaller than the bunch length.**
- Beam particles collide with the center of the other beam at their waist position.
- High beam-beam performance is expected, if **bunch population is enough high.**
- Satisfying the **conditions** is possible?

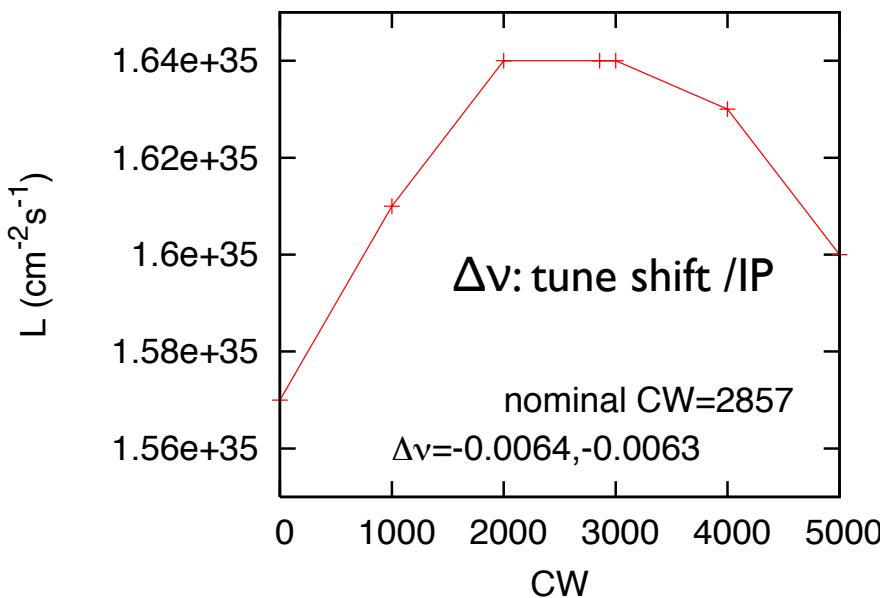
Luminosity with crab waist and/or crab cavity

10,Aug 2010, K. Ohmi

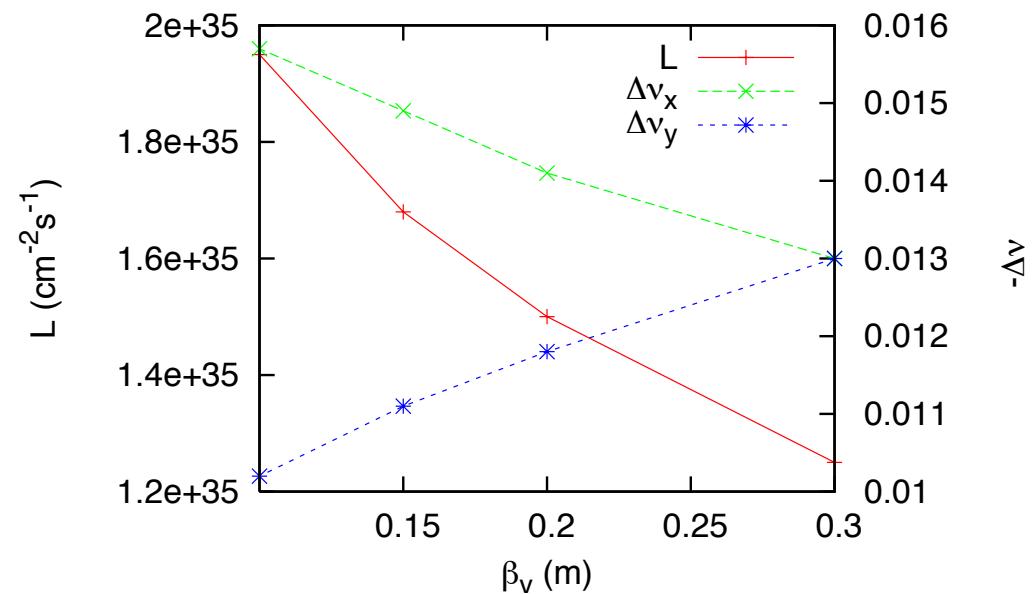
- $N_p = 4 \times 10^{11}$, $\beta_x = 0.3\text{m}$, $\beta_y = 0.075\text{m}$, $\varepsilon_N = 3.75\mu\text{m}$,
 $\theta = 315\mu\text{m}$, $\sigma_z = 11.8\text{cm}$, $\theta\sigma_z/2\sigma_x = 1.5$, $N_b = 1404$

Parameters by M. Zobov

Geometrical Luminosity for crab waist strength



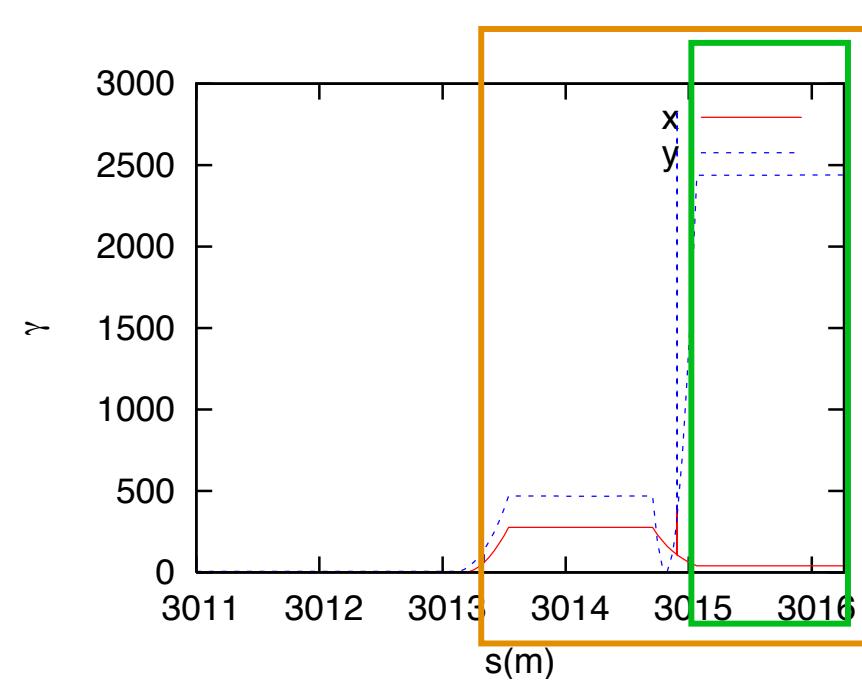
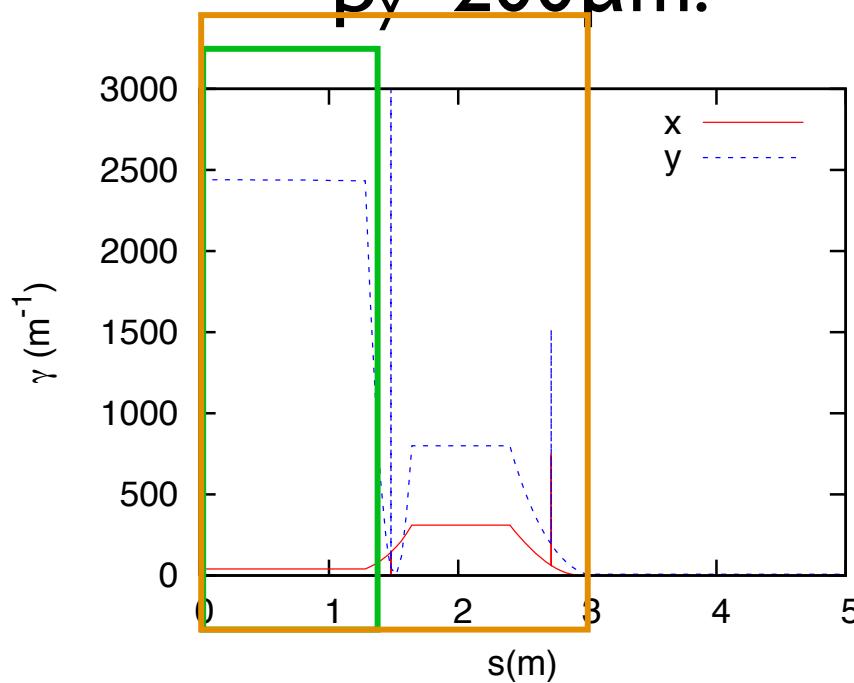
Geometrical luminosity and beam-beam parameter for crab cavity scheme with detuning β_y



Which challenge, low β or high beam-beam parameter?

Dynamic aperture issue for crab waist scheme in SuperKEKB

- IR nonlinearity, kinematic term and Quadrupole fringe, was dominant for the dynamic aperture in SuperB factories, because of the extremely small β function, $\beta_y=200\mu\text{m}$.



Kinematic nonlinearity

- Drift space at the interaction region (IR).
- Hamiltonian contain nonlinear term for p_x , p_y , δ .

$$H = (1 + \delta) - \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2}$$

- This nonlinearity is not negligible for very low beta IR.
 - ★ Chromaticity and its higher order
 - ★ Octupole and higher order nonlinearity

Chromaticity and nonlinearity

$$\begin{aligned} H_n &= (1 + \delta) - \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} - \frac{p_x^2 + p_y^2}{2} \\ &= -\frac{(p_x^2 + p_y^2)\delta}{2(1 + \delta)} + \frac{1}{8} \frac{(p_x^2 + p_y^2)^2}{(1 + \delta)^3} + \frac{1}{16} \frac{(p_x^2 + p_y^2)^4}{(1 + \delta)^5} + \dots \end{aligned}$$

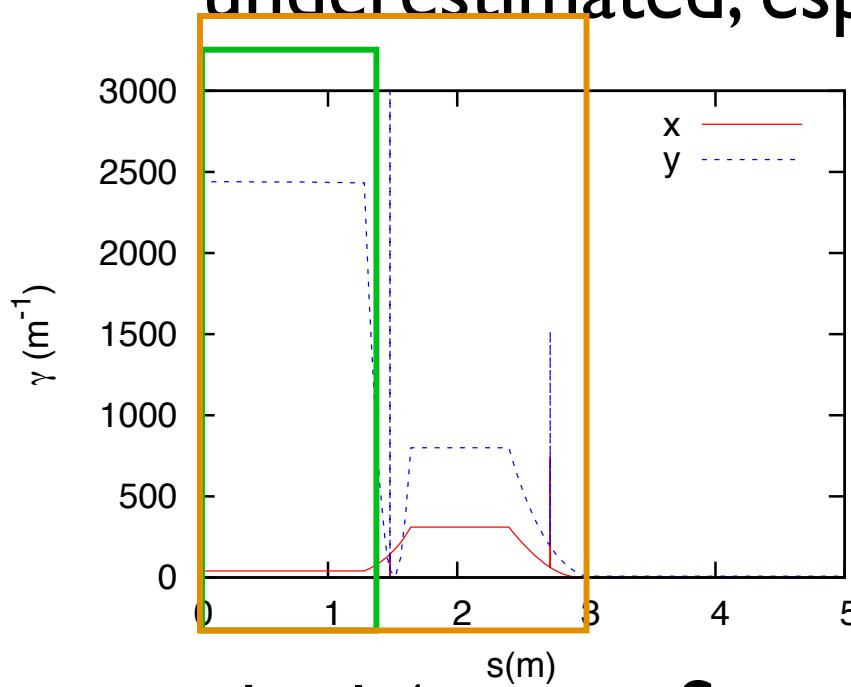
- Subtract linear (drift) motion
- First term gives chromaticity
- Second and later give octupole and higher nonlinearity

$$\sqrt{1 - x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128}$$

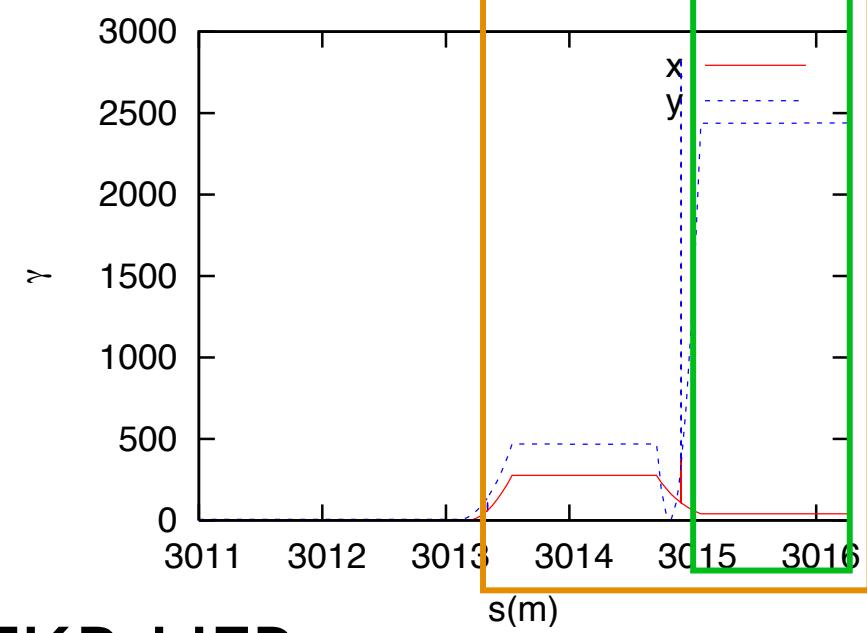
$$P^2 \sim \gamma J$$

$$\gamma = \frac{1 + \alpha^2}{\beta}$$

- These nonlinearities are strong at high γ region near IP.
- We consider L_0 area and QFI-QFI area in this presentation. It means that the effects are underestimated, especially for horizontal.



SuperKEKB-HER



Quadrupole edge nonlinearity

- Quadrupole nonlinearity at its face of IR

E. Forest

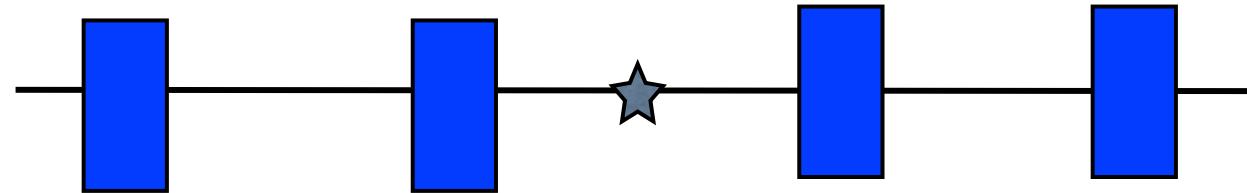
$$H_E = -\frac{k}{1 + \delta} \frac{yp_y(3x^2 + y^2) - xp_x(3y^2 + x^2)}{12}$$

$$k = \frac{eB'}{p_0 B} \quad \text{k} \sim 6 \text{m}^{-2}$$

Simple IR model

(like beam-beam simulation)

QFI-QFI area



$$\mathcal{M}_{IR} = e^{-H_{QF}} e^{-H_{L1}} e^{-H_{QD}} e^{-H_{L0}} e^{-H_{L0}} e^{-H_{QD}} e^{-H_{L1}} e^{-H_{QF}}$$

$$\mathcal{M}_{rev} = \mathcal{M}_{IR} \mathcal{M}_{arc}$$

- Arc is assumed to be linear.

$$\mathcal{M}_{arc} = M_{arc} = M_{IR}^{-1} M_0$$

- \mathbf{M}_{IR} is linear part of IR transfer matrix

$$M_{0,i=xyz} = \begin{pmatrix} \cos \mu_i + \alpha_i \sin \mu_i & \beta_i \sin \mu_i \\ -\gamma_i \sin \mu_i & \cos \mu_i - \alpha_i \sin \mu_i \end{pmatrix}$$

$$\mathcal{M}'_{IR} = \mathcal{M}_{IR} M_{IR}^{-1}$$

$$\mathcal{M}_{rev} = \mathcal{M}'_{IR} M_0$$

\mathcal{M}'_{IR} contains only nonlinear terms.

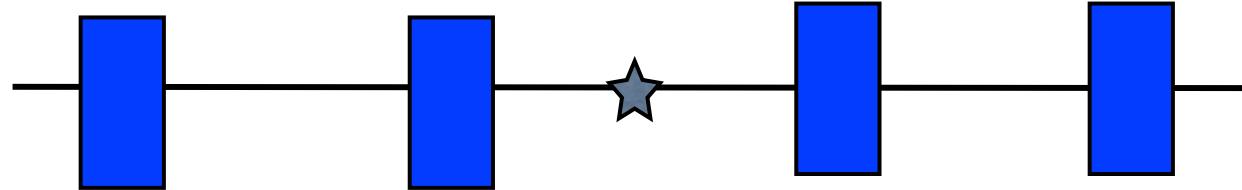
Crab waist

$$e^{-:H_K:L_0} e^{-:xp_y^2:/\theta} M_0 e^{:xp_y^2:/\theta} e^{-:H_K:L_0}$$
$$M_0 e^{:xp_y^2:/\theta} e^{-:H_K:L_0} e^{-:H_K:L_0} e^{-:xp_y^2:/\theta}$$

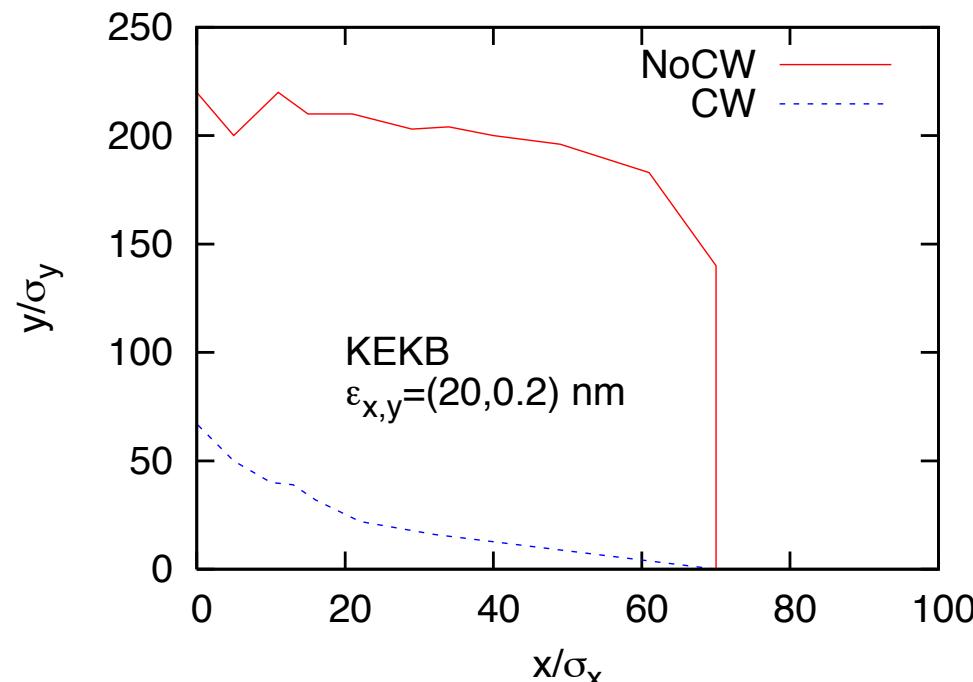
- If IR nonlinearity H_K is negligible, crab waist nonlinearity is cancelled outside of IR.
- IR nonlinearity breaks to cancel between the crab waist sextupoles.
- x^3 terms of the sextupole may affect something but are neglected now.
- The fact that the kinematic nonlinearity affect the aperture for crab waist scheme has been investigated by H. Koiso using SAD since several years ago.

On momentum aperture

QFI-QFI area



- KEKB (model) $\beta_x=0.6\text{m}$ $\beta_y=6\text{mm}$
- $L_0=1.3\text{m}$, $L_{QD}=2.3\text{m}$ $K_{QD}=-0.69\text{m}^{-1}$,
 $L_I=2.57\text{m}$, $L_{QF}=2.07\text{m}$, $K_{QF}=0.24\text{m}^{-1}$

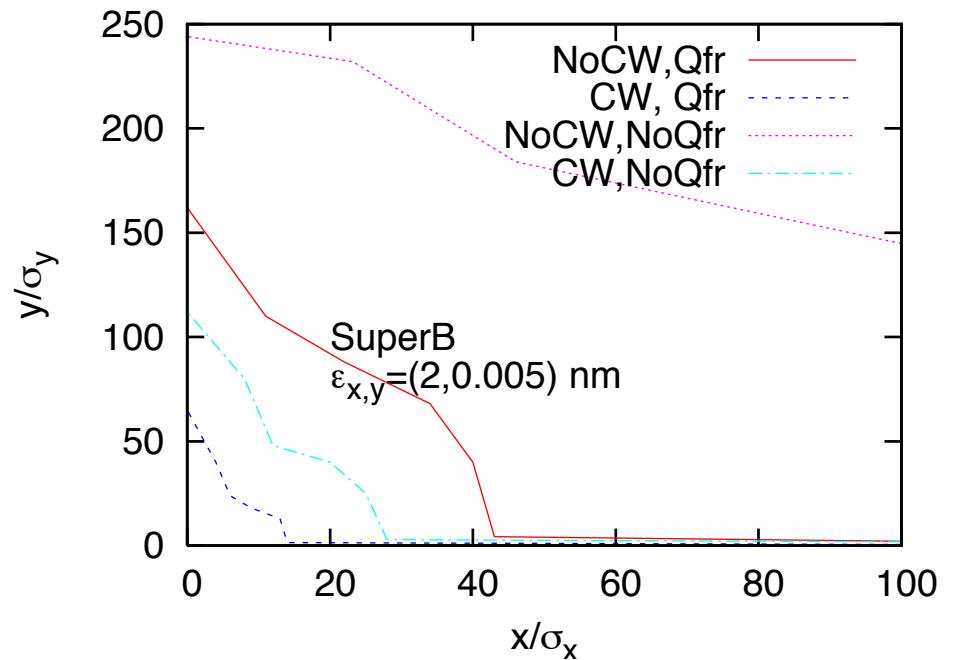
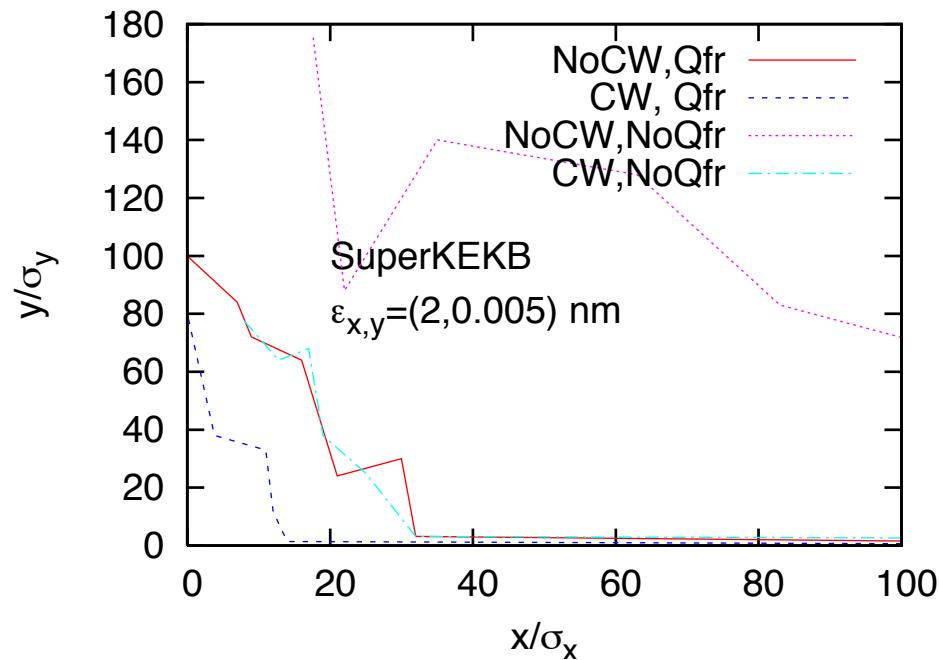


$$v_x=0.53, v_y=0.58$$

Agree with SAD
result by H. Koiso

On momentum aperture

- SuperKEKB, $L_0=0.73\text{m}$, $L_{QD}=0.39\text{m}$ $K_{QD}=-1.7\text{m}^{-1}$, $L_I=0.69\text{m}$,
 $L_{QF}=0.35\text{m}$, $K_{QF}=0.83\text{m}^{-1}$, $\beta_x=2\text{cm}$ $\beta_y=0.2\text{mm}$
- SuperB, $L_0=0.4\text{m}$, $L_{QD}=0.45\text{m}$ $K_{QD}=-2.7\text{m}^{-1}$, $L_I=0.4\text{m}$,
 $L_{QF}=0.20\text{m}$, $K_{QF}=1.2\text{m}^{-1}$, $\beta_x=2\text{cm}$ $\beta_y=0.2\text{mm}$



$$v_x=0.53, v_y=0.58$$

Chromaticity correction

$$\mathcal{M}'_{IR} = \prod_i^{IR} e^{-H_i} = \prod_i e^{H_{2,i}} e^{-H_{\xi,i}} e^{-H_{K,i}} e^{-H_{2,i}}$$

$e^{-H_{2,i}}$: Linear transformation IP to i-th element

$e^{-H_{\xi,i}}$: Chromatic terms, Quadratic term times a function of δ

$e^{-H_{K,i}}$: Higher order kinematic terms

I. Remove chromatic term element by element

$$\mathcal{M}''_{IR} = \prod_i e^{H_{2,i}} e^{-H_{K,i}} e^{-H_{2,i}}$$

Aperture is independent of δ .

This fact may be trivial.

More realistic chromaticity correction

Chromaticity correction outside of IR using H_c , quadratic term of x, p_x, y, p_y times $f(\delta)$.

$$e^{-H_{c,out}} \mathcal{M}'_{IR} e^{-H_{c,in}}$$

Choice of H_c

$$e^{H_{c,in(out)}} = \prod_i^{up(down)stream} e^{H_{2,i}} e^{-H_{\xi,i}} e^{-H_{2,i}}$$

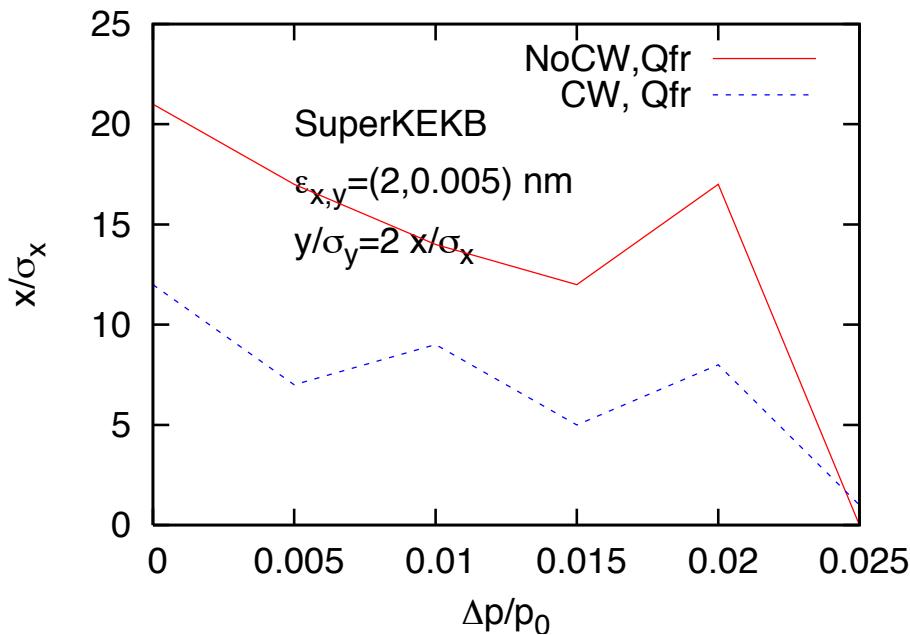
Chromaticity is corrected IP and outside of IR, but chromatic aberrations in IR section are remained.

Possible perfect chromaticity correction

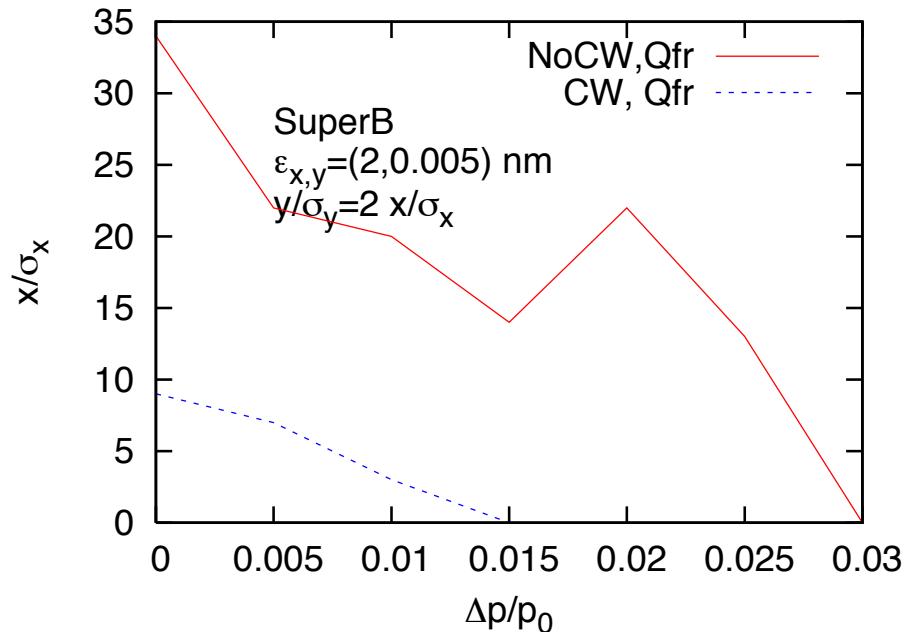
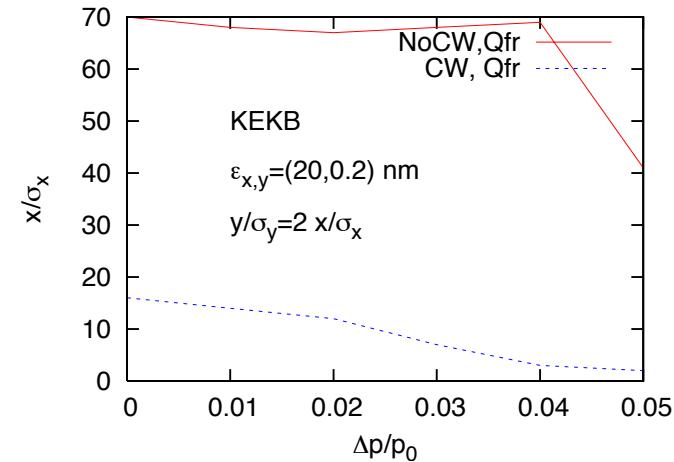
Off momentum aperture

- Synchrotron oscillation. $\Delta p/p_0$
- x-y oscillation, $y/\sigma_y = 2x/\sigma_x$.

$$v_x=0.53, v_y=0.58$$



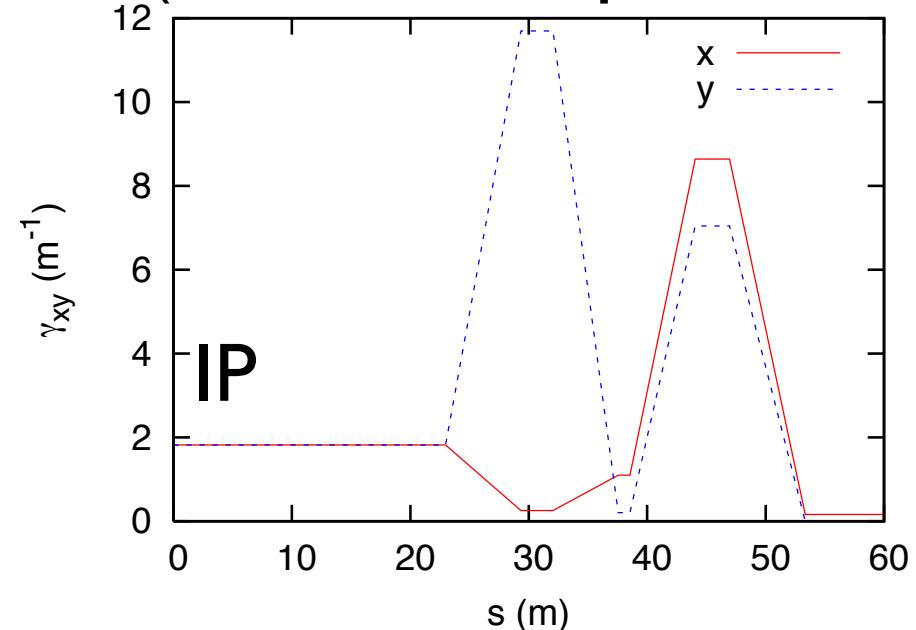
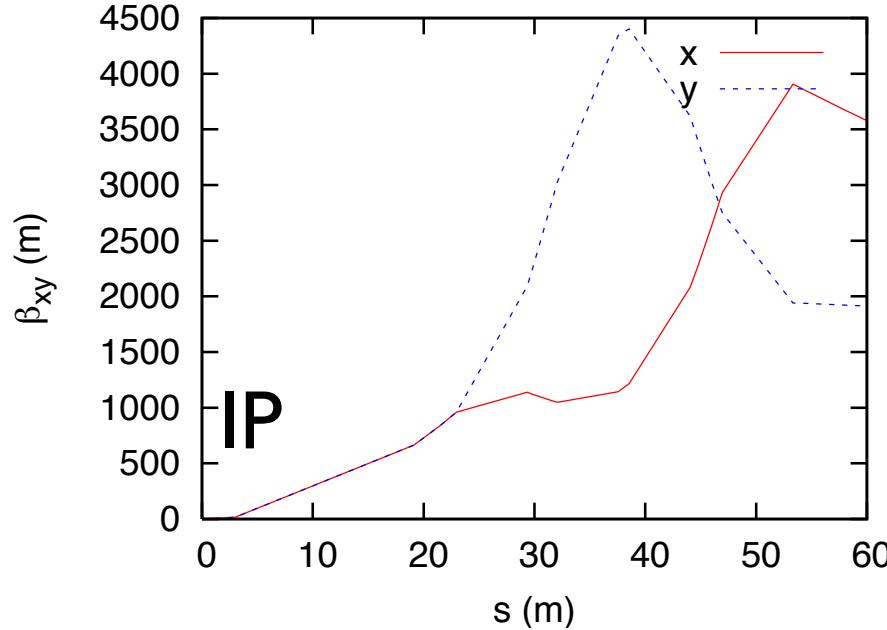
CW=10(nominal l/0.08=12.5)



CW=12.5(nominal l/0.05=20)

Dynamic aperture in Crab waist scheme in LHC

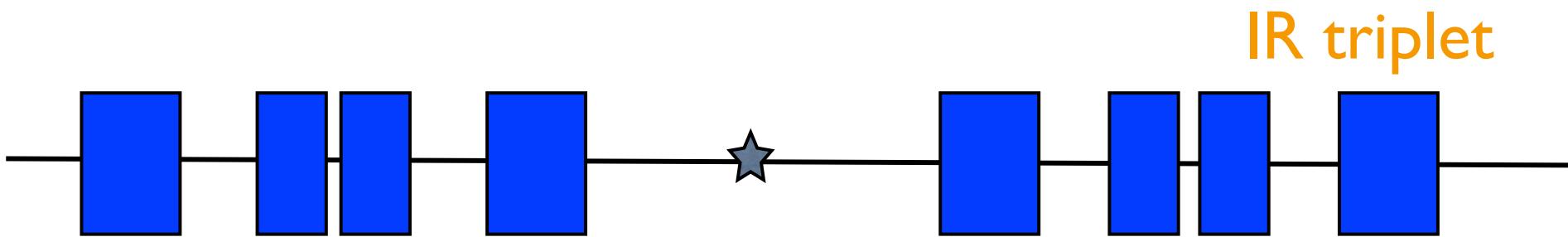
- γ is lower than that of Super B factories
- k is also weaker. $k=0.01\text{m}^{-2}$ ($\sim 6\text{m}^{-2}$ for Super KEKB)



Effect of Kinematic term and Quadrupole
fringe is weak.

Quadrupole nonlinearity in LHC

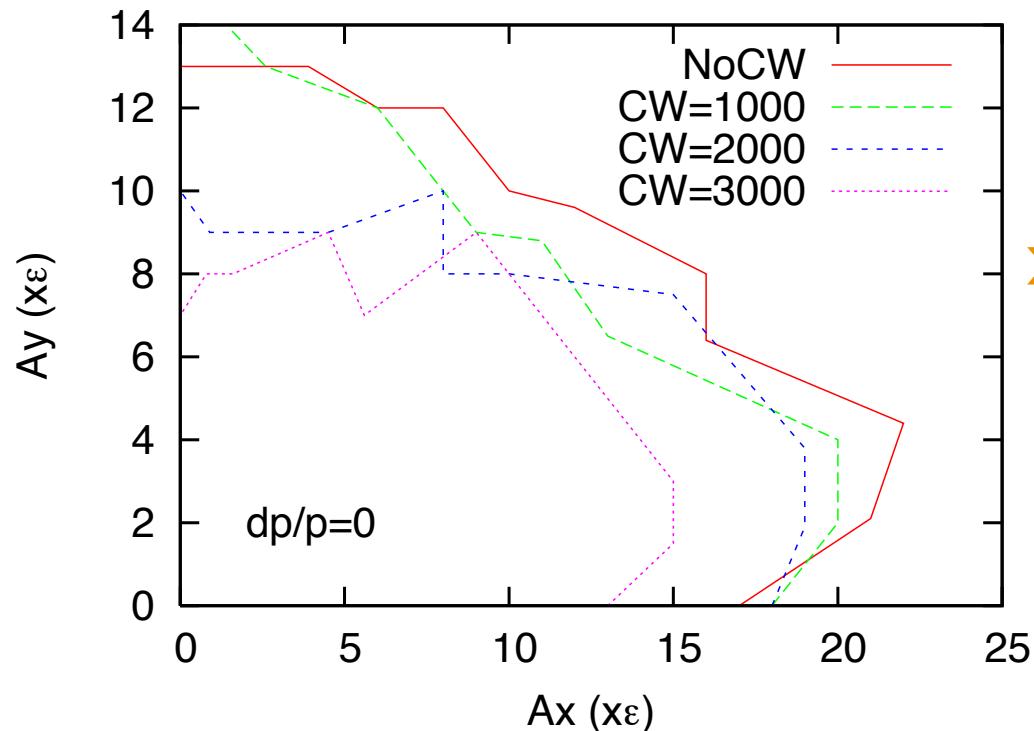
- Triplet of IPI, MQXA.IR(L)I, MQXB.A2R(L)I, MQXB.B2R(L)I, MQXA.3R(L)I
- Multipole components of these magnets are dominant for the limit of the dynamic aperture.
- Kinematic term and fringe field was negligible in LHC.
- Study with a model containing the 8 IR magnets.



Multipole table of Quadrupole is given by T. Rogelio.

On momentum aperture

- Degradation is seen for CW=3000, 300 μ rad crossing.



x^3 term of CW is not considered now.

Chromaticity correction, local or global

In super B factories, local chromaticity correction is adopted , because IR chromaticity is extremely large, and large beta function at IR is helpful for the chromaticity correction.

$$e^{-H_{c,out}} \mathcal{M}'_{IR} e^{-H_{c,in}}$$

local $e^{:xp_y^2:/\theta} e^{-H_{c,out}} \mathcal{M}'_{IR} e^{-H_{c,in}} e^{-:xp_y^2:/\theta}$

or

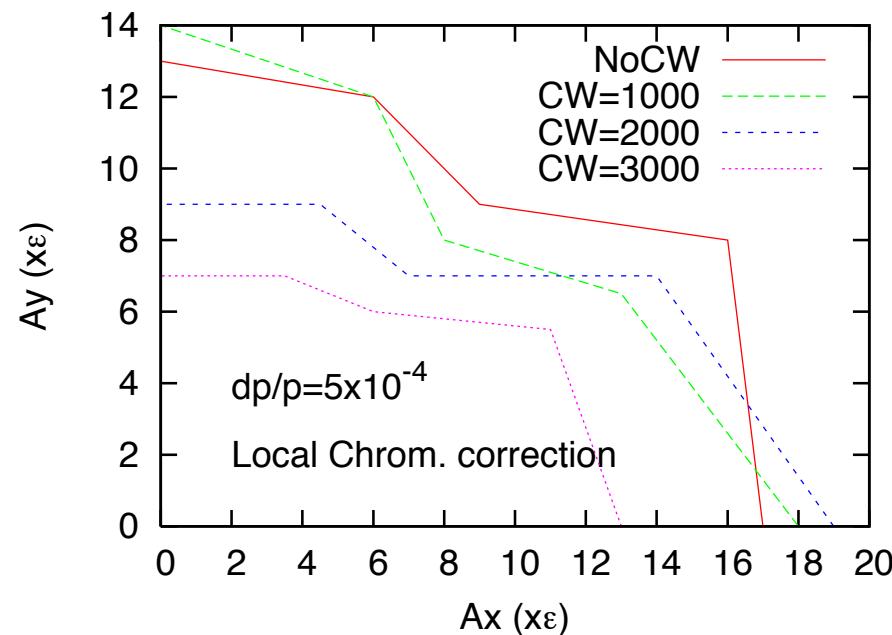
global $e^{-H_{c,out}} e^{:xp_y^2:/\theta} \mathcal{M}'_{IR} e^{-:xp_y^2:/\theta} e^{-H_{c,in}}$

Chromaticity correction of Hc is done inside or outside of the crab waist.

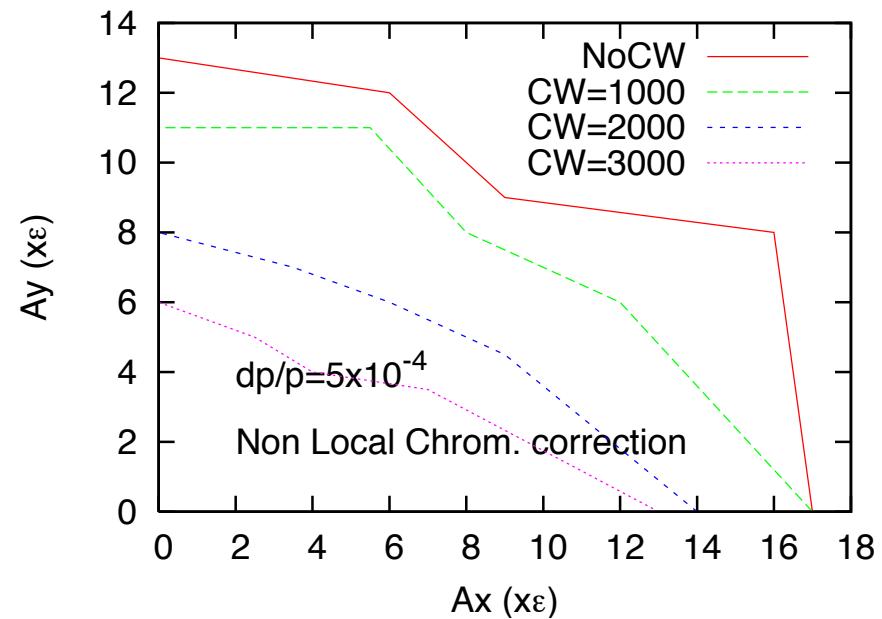
Off-momentum aperture

- $d\mathbf{p}/\mathbf{p} = 5 \times 10^{-4}$ where $\sigma_{d\mathbf{p}/\mathbf{p}} = 1 \times 10^{-4}$

local



global



Local chromaticity correction is better.

Summary

- Some simulations were carried out for HE-LHC.
- Beam-beam effect for ideal (HE-)LHC machine is weak.
- Effect of x-y coupling is weak in (HE-)LHC.
- Various collision scheme for HL-LHC have feasibility from the view of the beam-beam. Beam-beam parameter $\xi=0.03/\text{IP}$ is challengeable.
- Crab cavity studies are on-going favorably.
- In the crab waist scheme, local chromaticity and local nonlinearity corrections are required.

Summary II (the future)

- The strong-strong code is based on single IP collision. The weak-strong (BBWS) is extended two IP.
- The strong-strong code will be extended multi IP and including IR nonlinearity.



Oide-Kubo method in SAD

- Calculate $\langle p_i p_j \rangle$ under synchrotron radiation
- $\langle P_i P_j \rangle$ in the beam frame.

$$\langle P_i P_j \rangle = L_3 \langle p_i p_j \rangle L_3 \quad L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\gamma \end{pmatrix}$$

- Diagonalizing $\langle P_i P_j \rangle$, eigenvalues are obtained as u_1, u_2, u_3 .

$$R \langle P_i P_j \rangle R^t = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}$$

Calculate diffusion rate of the eigen-momentum

$$\frac{\Delta \langle w_1^2 \rangle}{\Delta t} = c_I [(g_2 - g_1) + (g_3 - g_1)]$$

$$\frac{\Delta \langle w_2^2 \rangle}{\Delta t} = c_I [(g_1 - g_2) + (g_3 - g_2)]$$

$$\frac{\Delta \langle w_3^2 \rangle}{\Delta t} = c_I [(g_1 - g_3) + (g_2 - g_3)]$$

$$c_I = \frac{r_e^2 N(\log)}{4\pi\gamma^3\varepsilon_1\varepsilon_2\varepsilon_3}$$

$$g_1 = \int_0^{\pi/2} \frac{2u_1 \sin^2 q \cos q dq}{\sqrt{(\sin^2 q + \frac{u_1}{u_2} \cos^2 q)(\sin^2 q + \frac{u_1}{u_3} \cos^2 q)}}$$

• where $g_2 = \int_0^{\pi/2} \frac{2u_2 \sin^2 q \cos q dq}{\sqrt{(\sin^2 q + \frac{u_2}{u_1} \cos^2 q)(\sin^2 q + \frac{u_2}{u_3} \cos^2 q)}}$

$$g_3 = \int_0^{\pi/2} \frac{2u_3 \sin^2 q \cos q dq}{\sqrt{(\sin^2 q + \frac{u_3}{u_1} \cos^2 q)(\sin^2 q + \frac{u_3}{u_2} \cos^2 q)}}$$

- Return to the physical coordinate in the lab frame.

$$\Delta\langle p_i p_j \rangle = L_3^{-1} R \begin{pmatrix} \Delta\langle w_1^2 \rangle & 0 & 0 \\ 0 & \Delta\langle w_2^2 \rangle & 0 \\ 0 & 0 & \Delta\langle w_3^2 \rangle \end{pmatrix} R^t L_3^{-1}$$

Solve beam envelope equation

Not used. It is enough to get the diffusion rate for single path issues

$$\langle x_i x_j \rangle = M(s) \langle x_i x_j \rangle M^t(s) + \int_s^{s+C} M(s+C, s_1) \Delta \langle x_i x_j \rangle M(s+C, s_1) ds_1$$

$$x_i = [x, p_x/p_0, y, p_y/p_0, z, (p_z - p_0)/p_0]$$

M: transfer matrix containing the radiation damping

- The IBS diffusion term $\Delta \langle p_i p_j \rangle$ is incorporated in $\Delta \langle x_i x_j \rangle$ together with the radiation excitation.
- $\Delta \langle p_i p_j \rangle$ is a function of $\langle x_i x_j \rangle$ or $\langle p_i p_j \rangle$, these processes are iterated.