Crab Cavity Voltage and Luminosity Calculation

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Abstract

The Luminosity is a very important parameter for every single accelerator on the world, because gives to us the interaction rate per unit cross section. So one of the goals of the desinger is treat to improve as muchs as possible the luminosity, but they are others factors which can affect and reduce it. The Crossing angle is a scheme whichs mitigates the effect of the Long-range beam-beam on the accelerator but also reduce the luminosity.One possible solution in order to keep low the Long-range beam-beam and improve the luminosity is the Crab Cavities (CC).

The CC is a superconducting RF cavity operated in a transverse dipole mode, which provides a transverse kick on the beam particles that varies with the longitudinal position along the bunch. The kick produces a rotation on the bunch in order to achieve a head-on collision and therefore increases the luminosity.

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Chapter 1

Crab Cavity Voltage

1.1 Introduction

To begin this study first calculate the energy gain for a particle which cross through an RF cavity. Consider a paticle q which pass for a cavity on the axis z with a speed v, and if assume an electric field $\vec{E}$ is the only accelerating mode [1]. The longitudinal $\vec{E}$ on the axis is

$$E_z(z,t) = \mathfrak{R}_\mathfrak{e}[\vec{E}\cdot\vec{z}e^{-j\omega_a}] \tag{1.1}$$

and the charge equation of motion is

$$z = v(t - t_0) \tag{1.2}$$

so using the Loretz Force's equation, the energy gain for the particle is

$$\begin{aligned} \Delta E = \; & q\int \mathfrak{R}_\mathfrak{e}[E_z(z)e^{-j\omega_a(\frac{z}{v}-t_0)}]dz \\ & q\int \mathfrak{R}_\mathfrak{e}[e^{-j\phi_0}E_z(z)e^{-j\omega_a\frac{z}{v}}]dz \end{aligned} \tag{1.3}$$

where $\phi_0 = \omega_a t_0$. If $E_z(z)$ is even function the equation (1.3), becomes

$$\Delta E = q \int E_z(z) \cos(\omega_a \frac{z}{v}) dz \tag{1.4}$$

The Panofsky-Wenzel theorem, yields a relation between the transverse kick ($\Delta p_\perp$) and the energy gain (ΔE) in the next way

$$\frac{h}{R} \frac{d\frac{\Delta p_\perp}{p_0}}{d\phi} = \nabla_\perp \frac{-\Delta E}{\beta^2 E_0} \tag{1.5}$$

where p_0 is the initial particle momentum, E_0 the initial particle energy, $\frac{h}{R\phi}$ is the longitudinal phase-space of the particle, $\nabla_\perp$ in the transverse gradient, β is the fraction between the speed of the particle and the speed of the ligth [2].

1.2 Crab Cavity

So now we can describe the horizontal transverse kick which produce the CC (Figure 1.1) like:

$$\Delta p_x = -\frac{qV \sin(\phi_s + \frac{\omega z}{c})}{E_s} \tag{1.6}$$

where q the particle charge, V the voltage of the CC, ϕ_s the synchronous phase of the CC, ω the angular frequency of the CC, z the longitudinal coordinate of the particle with respect to the bunch center, c the velocity of ligth and E_s the particle energy [3].

Now for the Figure 1.1, we have the next relation.

$$\tan(\frac{\phi}{2}) = \frac{x}{z} \approx \frac{\Delta x}{\Delta z} \approx \frac{dx}{dz} = x' \tag{1.7}$$

In the other hand there is an equation which relates x with Δp_x, the equation is

$$x = M_{12} \Delta p_x \tag{1.8}$$

where, M_{12} is an element of the Matrix M which relates the effect of x due to a Δp_x. The Matrix M show us how we can traslate between two point in the accelerator's lattice.

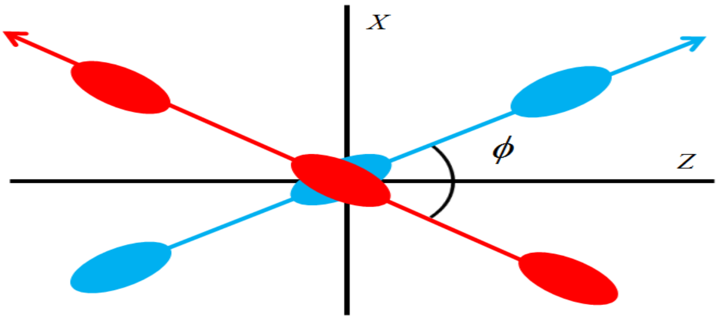

Figure 1.1: Scheme of the collision for a crossing angle.

Now replace the equation (1.6) on (1.7)

$$\begin{aligned}\tan(\tfrac{\phi}{2}) = & \frac{-d(M_{12}\frac{qV\sin(\phi_s+\frac{\omega z}{c})}{E_s})}{dz} \\ \approx & \frac{-d(M_{12}\frac{qV\omega z}{E_s c})}{dz} \\ = & M_{12}\frac{qV\omega}{Es} \end{aligned} \quad (1.9)$$

and if we manipulate to get V, the result is

$$V = \frac{cE_s\tan(\frac{\phi}{2})}{M_{12}q\omega} \quad (1.10)$$

Here we need to specify if we are using a Local Crab Cavity or a Global Crab Cavity.

1.2.1 Local Crab Cavity

Local Crab Cavity (LCC), in this case we put a crab cavity close to the interaction point (IP) in such way that the phase advance difference between the CC and the IP is close to $\frac{\pi}{2}$ and a symetric position with respect to the IP we introduce other CC whichs compensates the tranverse kick of the first one (Figure 1.2). So the effect of CC is just in a very specific region of the lattice (Figure 1.3).

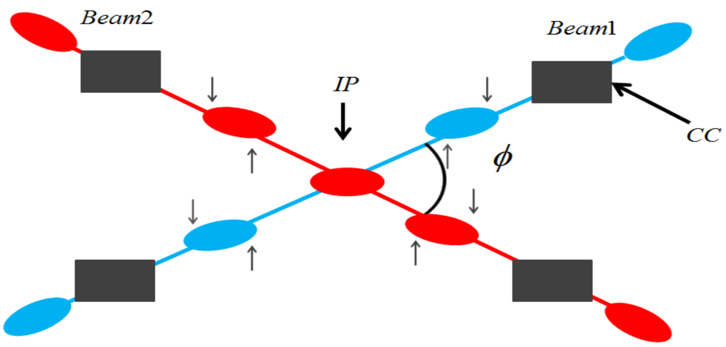

Figure 1.2: Scheme of the effect of the LCC in the horizontal axis of the bunch.

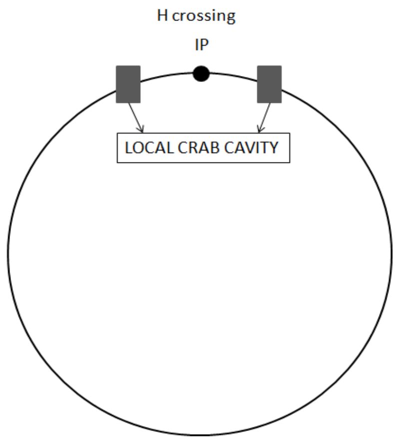

Figure 1.3: The LCC scheme in the lattice for a circular accelerator with a horizontal beam crabbed due to the CC.

In the case of the LCC the M is the normal Matrix of structure

$$\begin{bmatrix} x_2 \\ x'_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x'_1 \end{bmatrix} \tag{1.11}$$

where in (1.11), we have

$$
\begin{aligned}
M_{11} &= \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\Psi + \alpha_1 \sin\Delta\Psi) \\
M_{12} &= \sqrt{\beta_2\beta_1}\sin\Delta\Psi \\
M_{21} &= -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_2\beta_1}}\sin\Delta\Psi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_2\beta_1}}\cos\Delta\Psi \\
M_{22} &= \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\Psi - \alpha_2 \sin\Delta\Psi)
\end{aligned} \tag{1.12}
$$

where, $\beta_{1,2}$ is the beta fuction in the point (1,2), $\alpha_{1,2} \equiv -\frac{1}{2}\frac{d\beta_{1,2}}{ds}$ and $\Delta\Psi = \Delta\Psi_2 - \Delta\Psi_1$ is the phase advance between the point 1 and 2 [4].

The left CC has a voltage of

$$
V_L = \frac{cE_s \tan(\frac{\phi}{2})}{q\omega\sqrt{\beta_{IP}\beta_{CCL}}\sin\Delta\Psi_1} \tag{1.13}
$$

the rigth CC voltage is relate with the left CC voltage by (1.16)

$$
\begin{aligned}
V_R = & -M_{22}V_L \\
& = -M_{22}\frac{cE_s \tan(\frac{\phi}{2})}{q\omega\sqrt{\beta_{IP}\beta_{CCR}}\sin\Delta\Psi_2}
\end{aligned} \tag{1.14}
$$

where M'_{22} is the element (2,2) of the transfer matrix which traslate for the left CC to the rigth CC. Using (2.17) the equation 1.16 becomes

$$
V_R = -(\sqrt{\frac{\beta_{CCL}}{\beta_{CCR}}}\cos\Delta\Psi_{CC} - \alpha_{CCR}\sin\Delta\Psi_{CC})\frac{cE_s \tan(\frac{\phi}{2})}{q\omega\sqrt{\beta_{IP}\beta_{CCR}}\sin\Delta\Psi_2} \tag{1.15}
$$

where, c is the speed of the light, E_s the particle energy, α_{CCR} is the α function in the right CC, ϕ the crossing angle, ω the CC frequency, β_{IP} is the β funtion in the interacion point (IP),β_{CCL} is the β funtion in the left CC, β_{CCR} is the β funtion in the right CC, $\Delta\Psi_1$ is the difference of the phase advance between the left CC and the IP, $\Delta\Psi_2$ is the difference of the phase advance between the IP and the right CC and $\Delta\Psi_{CC}$ is the difference of the phase advance between the left CC and the right CC. In the ideal conditions $\Delta\Psi_{CC} = \pi$, so the equation (1.15) results

$$V_R = \frac{cE_s \tan(\frac{\phi}{2})}{q\omega\sqrt{\beta_{IP}\beta_{CCR}} \sin \Delta\Psi_2} \sqrt{\frac{\beta_{CCL}}{\beta_{CCR}}} \tag{1.16}$$

1.2.2 Global Crab Cavity

In the case of Global Crab Cavity, we put the CC in such way that $\cos(\Delta\Psi - \pi Q) \approx 1$, we dont introduce another CC which compensates the effect of first CC (Figure.1.4). So the effect of the transverse kick is in the entire lattice (Figure.1.5).

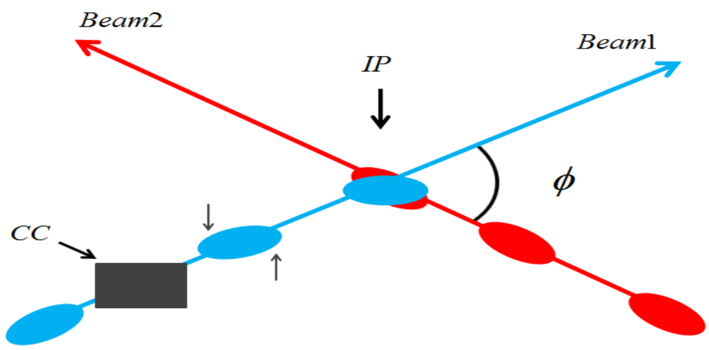

Figure 1.4: Scheme of the effect of the GCC in the horizontal axis of the bunch.

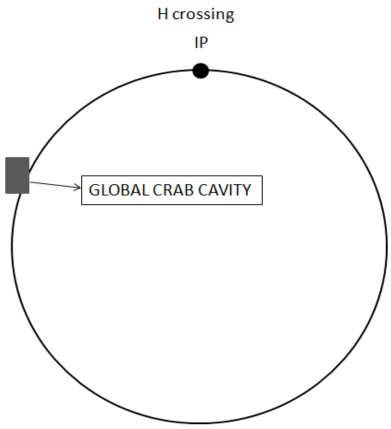

Figure 1.5: The LCC scheme in the lattice for a circular accelerator with a horizontal beam crabbed due to a CC.

So for the GCC we need to see how the transverse kick affect the bunch's trajectory when it pass again into the CC. We want that when the bunch makes a complete turn it has the same initial position, this can see like the effect of the translate Matrix plus the transverse kick gives to us the same initial position.

$$M \begin{bmatrix} x_1 \\ x_1' \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta p_x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1' \end{bmatrix} \tag{1.17}$$

So solving the equation (1.17) for x_1 and x_1' is

$$\begin{bmatrix} x_1 \\ x_1' \end{bmatrix} = (I - M)^{-1} \begin{bmatrix} x_1 \\ x_1' \end{bmatrix} \tag{1.18}$$

The matrix $(I - M)^{-1}$ can be recast using $M = e^{2\pi QJ}$ [4] with

$$J = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{bmatrix} \tag{1.19}$$

In this way we can get that

$$\begin{aligned} (I - M)^{-1} = & \ (I - e^{2\pi QJ})^{-1} \\ & = (e^{\pi QJ}(e^{-\pi QJ} - e^{\pi QJ}))^{-1} \\ & = -(2J \sin(\pi Q))^{-1}(e^{\pi QJ})^{-1} \\ & = \frac{Je^{-\pi QJ}}{2\sin(\pi Q)} \\ & = \frac{J\cos(\pi Q) + I\sin(\pi Q)}{2\sin(\pi Q)} \end{aligned} \tag{1.20}$$

so the equation the initial position will be

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_1' \end{bmatrix} = & \ \frac{1}{2\sin(\pi Q)}\{\cos(\pi Q) \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{bmatrix} + \sin(\pi Q) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\} \begin{bmatrix} 0 \\ \Delta p_x \end{bmatrix} \\ & = \frac{\Delta p_x}{2\sin(\pi Q)}\{\cos(\pi Q) \begin{bmatrix} \beta_1 \\ -\alpha_1 \end{bmatrix} + \sin(\pi Q) \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \\ & = \frac{\Delta p_x}{2\sin(\pi Q)} \begin{bmatrix} \beta_1 \cos(\pi Q) \\ \sin(\pi Q) - \alpha_1 \cos(\pi Q) \end{bmatrix} \end{aligned} \tag{1.21}$$

Now apply the traslate matrix to the initial conditions we get the effect of x due to a Δp_x

$$\begin{bmatrix} x_2 \\ x'_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x'_1 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \frac{\Delta p_x}{2\sin(\pi Q)} \begin{bmatrix} \beta_1 \cos(\pi Q) \\ \sin(\pi Q) - \alpha_1 \cos(\pi Q) \end{bmatrix} \tag{1.22}$$

making a little of algebra, we get

$$\begin{aligned} x = {} & \frac{\Delta p_x}{2\sin(\pi Q)}[\beta_1 \cos(\pi Q)\sqrt{\frac{\beta_2}{\beta_1}}\{\cos(\Delta\Psi) + \alpha_1 \sin(\Delta\Psi)\} + \{\sin(\pi Q) - \alpha_1 \cos(\pi Q)\}\sqrt{\beta_2\beta_1}\sin(\Delta\Psi \\ = {} & \frac{\Delta p_x}{2\sin(\pi Q)}\sqrt{\beta_1\beta_2}\{[\cos(\Delta\Psi) + \alpha_1 \sin(\Delta\Psi)]\cos(\pi Q) + [\sin(\pi Q) - \alpha_1 \cos(\pi Q)]\sin(\Delta\Psi)\} \\ = {} & \frac{\Delta p_x}{2\sin(\pi Q)}\sqrt{\beta_1\beta_2}\{\cos(\Delta\Psi)\cos(\pi Q) + \alpha_1 \sin(\Delta\Psi)\cos(\pi Q) + \sin(\pi Q)\sin(\Delta\Psi) - \alpha_1 \cos(\pi Q) \\ = {} & \frac{\Delta p_x}{2\sin(\pi Q)}\sqrt{\beta_1\beta_2}\cos(\Delta\Psi - \pi Q) \end{aligned}$$

For the equation (1.23) we have that for a GCC M_{12} is

$$M_{12} = \frac{\cos(\Delta\Psi - \pi Q)\sqrt{\beta_1\beta_2}}{2\sin(\pi Q)} \tag{1.24}$$

The voltage for a GCC is

$$\begin{aligned} V = {} & \frac{cE_s \tan(\frac{\phi}{2})}{q\omega} \frac{2\sin(\pi Q)}{\cos(\Delta\Psi - \pi Q)\sqrt{\beta_1\beta_2}} \\ = {} & \frac{cE_s \tan(\frac{\phi}{2})}{q\omega\sqrt{\beta_{IP}\beta_{CC}}} \frac{2\sin(\pi Q)}{\cos(\Delta\Psi - \pi Q)} \end{aligned} \tag{1.25}$$

where, c is the speed of the light, E_s the particle energy, ϕ the crossing angle, ω the CC frequency, β_{IP} is the β funtion in the interacion point (IP), β_{CC} is the β funtion in the CC and $\Delta\Psi$ is the difference of the phase advance between the CC and the IP.

The CC voltages are in agreement with the previous results [5].

Chapter 2

Luminosity Calculation

2.1 Introduction

For a two bunches which collides with the same number of particles N, the same crossing area A and they are moving in an opposite direction (Figure 2.1), any particle in each bunch will "see" a fraction of the area of the other bunch $\frac{N\sigma_p}{A}$ obscured by the interation cross section σ_p and if the frequency of bunch collision is f, we obtain the number of interaction per second (equation (2.1)) [4].

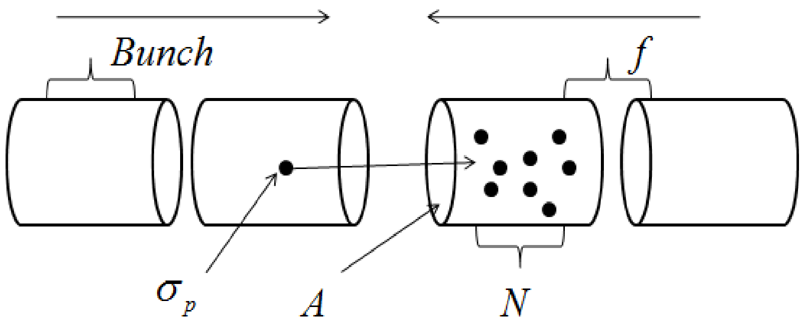

Figure 2.1: Scheme of the collision on two bunch.

$$\frac{dR}{dt} = f\frac{N\sigma_p}{A} \tag{2.1}$$

So we define the luminosity like the interaction rate per unit cross section (equation (2.2)).

$$\mathcal{L} = f\frac{N}{A} \tag{2.2}$$

and the units of the $\mathcal{L}$ are $cm^{-2}sec^{-1}$.

For two bunches with a certain distribution $\rho(x, y, z)$ the luminosity becomes

$$\mathcal{L} \propto KN_1N_2fN_b \int\int\int\int \rho_1(x, y, s - s_0)\rho_2(x, y, s - s_0)dxdydsds_0 \tag{2.3}$$

with the kinematic factor K (because the beams are colliding) is given by

$$K = \frac{\sqrt{(\vec{v_1} - \vec{v_2})^2 + (\vec{v_1} \times \vec{v_2})^2}}{|\vec{v_1}|\,|\vec{v_2}|} \tag{2.4}$$

If we can separate the distribution like

$$\rho_1(x, y, s - s_0) = \rho_{1x}(x)\rho_{1y}(y)\rho_{1s}(s - s_0) \tag{2.5}$$

the luminosity of two colliding beams can be written as

$$\mathcal{L} = KN_1N_2fN_b \int\int\int\int \rho_{x1}(x)\rho_{y1}(y)\rho_{s1}(s_1 - s_0)\rho_{x2}(x)\rho_{y2}(y)\rho_{s2}(s_2 - s_0)dxdydsds_0 \tag{2.6}$$

and assuming Gaussian distribution

$$\rho(z) = \frac{1}{\sigma_z\sqrt{(2\pi)}}e^{-\frac{z^2}{2\sigma_z^2}} \tag{2.7}$$

With this consideration , the equation (2.6) becomes

$$\mathcal{L} = \frac{2N_1N_2fN_b}{\sqrt{2\pi}^6\sigma_{x1}\sigma_{y1}\sigma_{s1}\sigma_{x2}\sigma_{y2}\sigma_{s2}} \int\int\int\int e^{\frac{-x1^2}{2\sigma_{x1}}} e^{\frac{-y1^2}{2\sigma_{y1}}} e^{\frac{-(s1-s0)^2}{2\sigma_{s1}}} e^{\frac{-x2^2}{2\sigma_{x2}}} e^{\frac{-y2^2}{2\sigma_{y2}}} e^{\frac{-(s2-s0)^2}{2\sigma_{s2}}} dxdydsds_0 \tag{2.8}$$

Here we made the Luminosity Calculation for three differents case:

CASE I : Head on Collision
CASE II : Crosing angle
CASE III: Crossing angle with a CC.

2.2 Head on Collision.

The head on collision is the simplest case. They are two bunches moving in the oppositive direction (Figure 2.2).

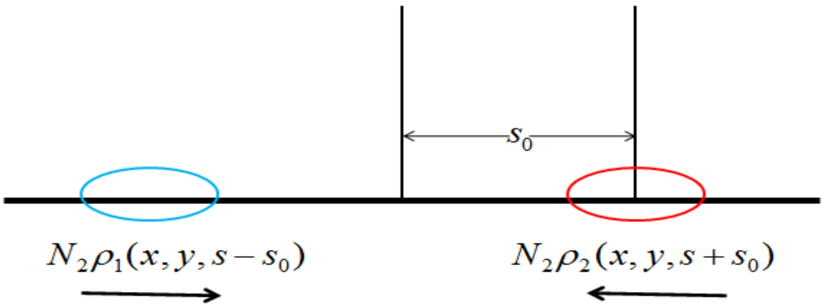

Figure 2.2: Scheme of the head-on collision on two bunch.

We see different cases.

1) First we assume that

$$\sigma_{s1} = \sigma_{s2};\ \sigma_{x1} = \sigma_{x2};\ \sigma_{y1} = \sigma_{y2};\ \sigma_{x1} \neq \sigma_{y1};\ \sigma_{x1} \neq \sigma_{y2}.\ \vec{v_1} = -\vec{v_2} \text{ so } K = 2$$

so in the case of $\sigma_1 = \sigma_2$ we have $\rho_1\rho_2 = \rho^2$. In this way the equation (2.8) becomes

$$\mathcal{L} = \frac{2N_1N_2fN_b}{\sqrt{2\pi}^6\sigma_x^2\sigma_y^2\sigma_s^2}\int\int\int\int e^{\frac{-x^2}{\sigma_x^2}}e^{\frac{-y^2}{\sigma_y^2}}e^{\frac{-s^2}{\sigma_s^2}}e^{\frac{-s_0^2}{\sigma_s^2}}dxdydsds_0 \tag{2.9}$$

we know that

$$\int e^{-at^2} = \sqrt{\frac{\pi}{a}} \tag{2.10}$$

$$\frac{1}{2\pi\sigma_y^2}\int e^{-\frac{1}{\sigma_y^2}}dy = \frac{1}{2\sqrt{\pi}\sigma_y} \tag{2.11}$$

that's means for each integral on equation (2.9) we have a term like (2.11), so the equation (2.9) results

$$\mathcal{L} = \frac{N_1N_2fN_b}{4\pi\sigma_x\sigma_y} \tag{2.12}$$

2) Second we have that

$\sigma_{s1} \approx \sigma_{s2}$, but $\sigma_{x1} \neq \sigma_{x2}$ and $\sigma_{y1} \neq \sigma_{y2}$.

So the equation (2.8) is

$$\mathcal{L} = \frac{2N_1 N_2 f N_b}{\sqrt{2\pi}^6 \sigma_{x1}\sigma_{x2}\sigma_{y1}\sigma_{y2}\sigma_s^2} \int\int\int\int e^{\frac{-x^2}{2}\frac{\sigma_{x1}^2+\sigma_{x2}^2}{\sigma_{x1}^2\sigma_{x2}^2}} e^{\frac{-y^2}{2}\frac{\sigma_{y1}^2+\sigma_{y2}^2}{\sigma_{y1}^2\sigma_{y2}^2}} e^{\frac{-s^2}{\sigma_s^2}} e^{\frac{-s_0^2}{\sigma_s^2}} dxdydsds_0 \quad (2.13)$$

The firts integral has the result of

$$\frac{1}{2\pi\sigma_{y1}\sigma_{y2}} \int e^{\frac{-y^2}{2}\frac{\sigma_{y1}^2+\sigma_{y2}^2}{\sigma_{y1}^2\sigma_{y2}^2}} dy = \frac{1}{2\pi\sigma_{y1}\sigma_{y2}} \sqrt{\frac{2\pi\sigma_{y1}^2\sigma_{y2}^2}{\sigma_{y1}^2+\sigma_{y2}^2}} \quad (2.14)$$

making a little of algebra on the equation (2.14), we have

$$\frac{1}{2\pi\sigma_{y1}\sigma_{y2}} \int e^{\frac{-y^2}{2}\frac{\sigma_{y1}^2+\sigma_{y2}^2}{\sigma_{y1}^2\sigma_{y2}^2}} dy = \frac{1}{\sqrt{2\pi(\sigma_{y1}^2+\sigma_{y2}^2)}} \quad (2.15)$$

for the last two term in the equation (2.13) have the same value of equation (2.11), we obtain that the equation (2.13) is

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{2\pi\sqrt{\sigma_{x1}^2+\sigma_{x2}^2}\sqrt{\sigma_{y1}^2+\sigma_{y2}^2}} \quad (2.16)$$

2.3 Crossing Angle.

For the collision of two bunches which makes an angle when they collide (Figure 2.3). This scheme is knows like the crossing angle (Figure 2.4).

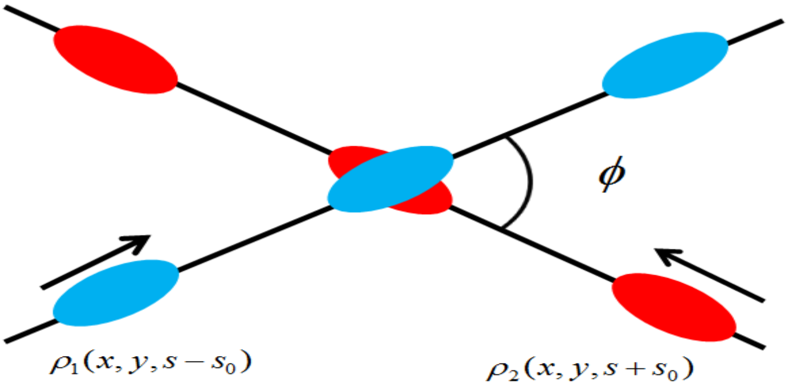

Figure 2.3: Scheme of the collision on two bunch with crossing angles.

$$\begin{aligned} x_1 &= x\cos(\frac{\phi}{2}) - s\sin(\frac{\phi}{2}) \\ s_1 &= s\cos(\frac{\phi}{2}) + x\sin(\frac{\phi}{2}) \\ x_2 &= x\cos(\frac{\phi}{2}) + s\sin(\frac{\phi}{2}) \\ s_2 &= s\cos(\frac{\phi}{2}) - x\sin(\frac{\phi}{2}) \end{aligned} \tag{2.17}$$

We assume that $|\vec{v_1}| = |\vec{v_2}| = v$, so the equation 2.4

$$K = \frac{\sqrt{4v^2\cos^2(\frac{\phi}{2}) - 4v^2\cos^2(\frac{\phi}{2})\sin^2(\frac{\phi}{2})}}{v^2} = \frac{2v^2\cos(\frac{\phi}{2})\sqrt{1 - sin^2(\frac{\phi}{2})}}{v^2} = 2\cos^2(\frac{\phi}{2}) \tag{2.18}$$

Therefore, with crossing angle the luminosity integral is given by

$$\mathcal{L} = 2\cos^2(\frac{\phi}{2})N_1N_2fN_b\int\int\int\int\rho_1(x_1)\rho_1(y)\rho_1(s_1 - s_0)\rho_2(x_2)\rho_2(y)\rho_2(s_2 + s_0)dxdydsds_0 \tag{2.19}$$

We use a Gaussian distribution (2.7) and we assume that $\sigma_{s1} = \sigma_{s2}$; $\sigma_{x1} = \sigma_{x2}$; $\sigma_{y1} = \sigma_{y2}$; $\sigma_{x1} \neq \sigma_{y1}$ and $\sigma_{x2} \neq \sigma_{y2}$, so the equation (2.19) becomes

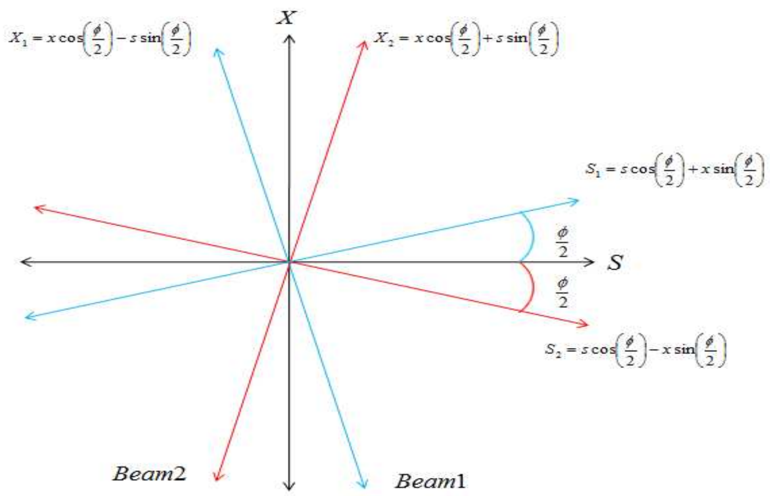

Figure 2.4: Axis scheme of the collision on two bunch with crossing angles

$$\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2}) N_1 N_2 f N_b}{\sqrt{2\pi}^6 \sigma_x^2 \sigma_y^2 \sigma_s^2} \int\int\int\int e^{\frac{-(x_1)^2}{2\sigma_x^2}} e^{\frac{-(x_2)^2}{2\sigma_x^2}} e^{\frac{-(s_1-s_0)^2}{2\sigma_s^2}} e^{\frac{-(s_2+s_0)^2}{2\sigma_x^2}} e^{\frac{-y^2}{\sigma_y^2}} dxdydsds_0 \quad (2.20)$$

$$\begin{aligned}
x_1^2 &= (x\cos(\frac{\phi}{2}) - s\sin(\frac{\phi}{2}))^2 \\
&= x^2\cos^2(\frac{\phi}{2}) - 2xs\cos(\frac{\phi}{2})\sin(\frac{\phi}{2}) + s^2\sin^2(\frac{\phi}{2}) \\
(s_1 - s_0)^2 &= (s\cos(\frac{\phi}{2}) + x\sin(\frac{\phi}{2}) - s_0)^2 \\
&= s^2\cos^2(\frac{\phi}{2}) + 2sx\cos(\frac{\phi}{2})\sin(\frac{\phi}{2}) + x^2\sin^2(\frac{\phi}{2}) - 2s_0(s\cos(\frac{\phi}{2}) + x\sin(\frac{\phi}{2})) + s_0^2 \\
x_2^2 &= (x\cos(\frac{\phi}{2}) + s\sin(\frac{\phi}{2}))^2 \\
&= x^2\cos^2(\frac{\phi}{2}) + 2sx\cos(\frac{\phi}{2})\sin(\frac{\phi}{2}) + s^2\sin^2(\frac{\phi}{2}) \\
(s_2 + s_0)^2 &= (s\cos(\frac{\phi}{2}) - x\sin(\frac{\phi}{2}) + s_0)^2 \\
&= s^2\cos^2(\frac{\phi}{2}) - 2sx\cos(\frac{\phi}{2})\sin(\frac{\phi}{2}) + x^2\sin^2(\frac{\phi}{2}) + 2s_0(s\cos(\frac{\phi}{2}) - x\sin(\frac{\phi}{2})) + s_0^2
\end{aligned} \quad (2.21)$$

The integral over y is very simple we use (2.11) and for the other variables we use we used the fact of

$$\int e^{-(at^2+bt+c)} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2-4ac}{4a}} \quad (2.22)$$

making an arrange the equation (2.20) becomes

$$\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\int\int\int e^{-(a_xx^2+b_xx+c_x)}e^{-(a_ss^2+b_ss+c_s)}e^{-(a_{s_0}s_0^2+b_{s_0}s_0+c_{s_0})}dxdsds_0 \quad (2.23)$$

with

$$\begin{aligned} a_x &= \frac{\cos^2\frac{\phi}{2}}{\sigma_x^2}+\frac{\sin^2\frac{\phi}{2}}{\sigma_s^2} \\ b_x &= \frac{2s_0\sin\frac{\phi}{2}}{\sigma_s^2} \\ c_x &= 0 \\ a_s &= \frac{\sin^2\frac{\phi}{2}}{\sigma_x^2}+\frac{\cos^2\frac{\phi}{2}}{\sigma_s^2} \\ c_s &= 0 \\ a_{s_0} &= \frac{2}{\sigma_s^2} \\ b_{s_0} &= 0 \\ c_{s_0} &= 0 \end{aligned} \quad (2.24)$$

Using (2.22), first we integrate with respect to x (here will assume that $\frac{\sin^2\frac{\phi}{2}}{\sigma_s^2}\approx 0$ so $a_x\approx\frac{\cos^2\frac{\phi}{2}}{\sigma_x^2}$) on (2.23).

$$\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\int\int\sqrt{\frac{\pi\sigma_x^2}{\cos^2\frac{\phi}{2}}}e^{-(s_0^2a_1)}e^{-(a_ss^2+b_ss+c_s)}e^{-(a_{s_0}s_0^2+b_{s_0}s_0+c_{s_0})}dsds_0 \quad (2.25)$$

where $a_1=\frac{b_x^2}{4a_x}=\frac{(2s_0\sin(\frac{\phi}{2}))^2}{\sigma_s^2}\frac{\sigma_x^2}{4\cos^2(\frac{\phi}{2})}\approx\sin^2(\frac{\phi}{2})\sigma_x^2\approx 0$

So for the values of a_1, a_s, b_s, c_s, a_{s_0}, b_{s_0} and c_{s_0} (2.25) results

$$\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\int\int\sqrt{\frac{\pi\sigma_x^2}{\cos^2\frac{\phi}{2}}}e^{-(\frac{\sigma_s^2\sin^2(\frac{\phi}{2})+\sigma_x^2\cos^2(\frac{\phi}{2})}{\sigma_x^2\sigma_s^2})s^2}e^{-\frac{s_0^2}{\sigma_s^2}}dsds_0 \quad (2.26)$$

No apply (2.11) in (2.26) results

$$\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\sqrt{\frac{\pi\sigma_x^2}{\cos^2\frac{\phi}{2}}}\sqrt{\frac{\pi\sigma_x^2\sigma_s^2}{\sigma_s^2\sin^2(\frac{\phi}{2})+\sigma_x^2\cos^2(\frac{\phi}{2})}}\sqrt{\pi\sigma_s^2} \quad (2.27)$$

making a little of algebra

$$\mathcal{L} = \frac{\cos{(\frac{\phi}{2})} N_1 N_2 f N_b}{4\pi\sigma_y} \frac{1}{\sqrt{\sigma_s^2 \sin^2(\frac{\phi}{2}) + \sigma_x^2 \cos^2(\frac{\phi}{2})}} \tag{2.28}$$

If we factored $\sigma_x^2 \cos^2(\frac{\phi}{2})$ into the denominator we have

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{4\pi\sigma_y\sigma_x} \frac{1}{\sqrt{1 + \frac{\sigma_s^2 \sin^2(\frac{\phi}{2})}{\sigma_x^2 \cos^2(\frac{\phi}{2})}}} \tag{2.29}$$

we assume for small ϕ, we have $\tan(\frac{\phi}{2}) \approx \sin(\frac{\phi}{2}) \approx \frac{\phi}{2}$.

The equation 2.29 results

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{4\pi\sigma_y\sigma_x} \frac{1}{\sqrt{1 + (\frac{\sigma_s}{\sigma_x}\frac{\phi}{2})^2}} \tag{2.30}$$

$$\mathcal{L} = \frac{N_1 N_2 f N_b}{4\pi\sigma_y\sigma_x} \frac{1}{\sqrt{1 + \phi_{piw}}} \tag{2.31}$$

where $\phi_{piw} = \frac{\sigma_s}{\sigma_x}\frac{\phi}{2}$ is the Piwinski angle.

2.4 Crosing Angle with a CC.

The CC is a superconducting RF cavity operated in a transverse dipole mode, which provides a transverse kick on the beam particles that varies with the longitudinal position along the bunch center (equation (2.32)). This effect can produce a rotation in the beam with a crossing angle $\frac{\phi}{2}$, so that produce the head-on collision.

$$\Delta p_x = -\frac{qV\sin(\phi_s + \frac{\omega z}{c})}{E_s} \tag{2.32}$$

where q the particle charge, V the voltage of the CC, ϕ_s the synchronous phase of the CC, ω the angular frequency of the CC, z the longitudinal coordinate of the particle with respect to the bunch center, c the velocity of ligth and E_s the particle energy [3].

The expresion for the horizontal coordinate at the IP (interacion point) X, can be express,

$$X = M_{12}\Delta p_x + X_0 = -M_{12}\frac{qV\sin(\phi_s + \frac{\omega z}{c})}{E_s} + X_o \tag{2.33}$$

where X_0 is the value o X without effect of the CC. The voltage of the CC is

$$V = \frac{cE_s\tan(\frac{\phi}{2})}{q\omega M_{12}} \tag{2.34}$$

replace equation (2.34) in (2.33) we have,

$$X = -M_{12}\frac{q\sin(\phi_s + \frac{\omega z}{c})}{E_s}\frac{cE_s\tan(\frac{\phi}{2})}{q\omega M_{12}} + X_0 = \frac{\tan(\frac{\phi}{2})}{\omega}\sin(\phi_s + \frac{\omega z}{c}) + X_0 \tag{2.35}$$

where we redefined $\phi_s + \frac{\omega z}{c} = K_{cr}(s \pm ct)$ with $K_{cr} = \frac{2\pi f_{crab}}{c} = \frac{\omega}{c}$. The change in the horizontal coordinate produces for the CC ($\Delta X_{1,2}$) is,

$$\Delta X_{1,2} = \pm\frac{\tan(\frac{\phi}{2})}{\omega}\sin(K_{cr}(s \mp ct)) \approx \pm\frac{\sin(\frac{\phi}{2})}{k_{cr}}\sin(K_{cr}(s \mp ct)) \tag{2.36}$$

2.4.1 Crosing Angle with a CC for each Beam.

In this case we have one CC for each beam.

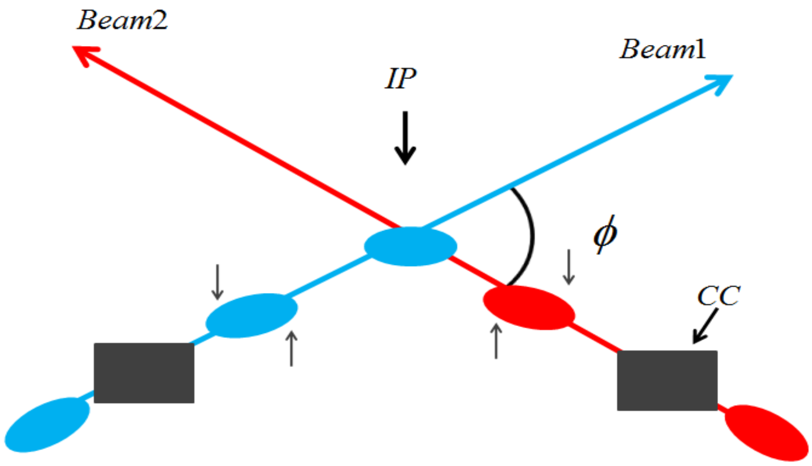

Figure 2.5: Scheme of the collision on two bunch with crossing angles and with two CC

The coordinates are,

$$\begin{array}{l} x_1 = x\cos(\frac{\phi}{2}) - s\sin(\frac{\phi}{2}) + \frac{1}{K_{cr}}\sin(K_{cr}(s-s_0))\sin(\frac{\phi}{2}) \\ s_1 = s\cos(\frac{\phi}{2}) + x\sin(\frac{\phi}{2}) \\ x_2 = x\cos(\frac{\phi}{2}) + s\sin(\frac{\phi}{2}) - \frac{1}{K_{cr}}\sin(K_{cr}(s+s_0))\sin(\frac{\phi}{2}) \\ s_2 = s\cos(\frac{\phi}{2}) - x\sin(\frac{\phi}{2}) \end{array} \tag{2.37}$$

with $K_{cr} = \frac{2\pi f_{crab}}{c}$, so with those definitions we have the same equation of (2.20)

$$\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^6\sigma_x^2\sigma_y^2\sigma_s^2}\int\int\int\int e^{\frac{-(x_1)^2}{2\sigma_x^2}}e^{\frac{-(x_2)^2}{2\sigma_x^2}}e^{\frac{-(s_1-s_0)^2}{2\sigma_s^2}}e^{\frac{-(s_2+s_0)^2}{2\sigma_x^2}}e^{\frac{-y^2}{\sigma_y^2}}\,dxdydsds_0 \tag{2.38}$$

For the moment we call

$$d_1 = \frac{1}{K_{cr}}\sin(K_{cr}(s-s_0))\sin(\frac{\phi}{2}) \tag{2.39}$$

and

$$d_2 = -\frac{1}{K_{cr}}\sin(K_{cr}(s+s_0))\sin(\frac{\phi}{2}) \tag{2.40}$$

$$
\begin{aligned}
&x_1^2 = (x\cos(\tfrac{\phi}{2}) - s\sin(\tfrac{\phi}{2}) + d_1)^2 = x^2\cos^2(\tfrac{\phi}{2}) - 2xs\cos(\tfrac{\phi}{2})\sin(\tfrac{\phi}{2}) + s^2\sin^2(\tfrac{\phi}{2})\\
&+d_1^2 + 2d_1(x\cos(\tfrac{\phi}{2}) - s\sin(\tfrac{\phi}{2}))\\
&(s_1 - s_0)^2 = (s\cos(\tfrac{\phi}{2}) + x\sin(\tfrac{\phi}{2}) - s_0)^2 = s^2\cos^2(\tfrac{\phi}{2}) + 2sx\cos(\tfrac{\phi}{2})\sin(\tfrac{\phi}{2}) + x^2\sin^2(\tfrac{\phi}{2})\\
&-2s_0(s\cos(\tfrac{\phi}{2}) + x\sin(\tfrac{\phi}{2})) + s_0^2\\
&x_2^2 = (x\cos(\tfrac{\phi}{2}) + s\sin(\tfrac{\phi}{2}) + d_2)^2 = x^2\cos^2(\tfrac{\phi}{2}) + 2sx\cos(\tfrac{\phi}{2}) + s^2\sin^2(\tfrac{\phi}{2})\\
&+d_2^2 + 2d_2(x\cos(\tfrac{\phi}{2}) + s\sin(\tfrac{\phi}{2}))\\
&(s_2 + s_0)^2 = (s\cos(\tfrac{\phi}{2}) - x\sin(\tfrac{\phi}{2}) + s_0)^2 = s^2\cos^2(\tfrac{\phi}{2}) - 2sx\cos(\tfrac{\phi}{2})\sin(\tfrac{\phi}{2}) + x^2\sin^2(\tfrac{\phi}{2})\\
&+2s_0(s\cos(\tfrac{\phi}{2}) - x\sin(\tfrac{\phi}{2})) + s_0^2
\end{aligned} \tag{2.41}
$$

now we write the equation (2.38) in order to use the formula (2.22), we get the same equation of (2.23)

$$
\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\int\int\int e^{-(a_xx^2+b_xx+c_x)}e^{-(a_ss^2+b_ss+c_s)}e^{-(a_{s_0}s_0^2+b_{s_0}s_0+c_{s_0})}e^{\frac{-(d_1^2+d_2^2)}{2\sigma_x^2}}dxdsds_0 \tag{2.42}
$$

with

$$
\begin{aligned}
a_x &= \tfrac{\cos^2\frac{\phi}{2}}{\sigma_x^2} + \tfrac{\sin^2\frac{\phi}{2}}{\sigma_s^2}\\
b_x &= \tfrac{(d_1+d_2)\cos(\frac{\phi}{2})}{\sigma_x^2} - \tfrac{2s_0\sin\frac{\phi}{2}}{\sigma_s^2}\\
c_x &= 0\\
a_s &= \tfrac{\sin^2\frac{\phi}{2}}{\sigma_x^2} + \tfrac{\cos^2\frac{\phi}{2}}{\sigma_s^2}\\
b_s &= \tfrac{(d_2-d_1)\sin(\frac{\phi}{2})}{\sigma_x^2}\\
c_s &= 0\\
a_{s_0} &= \tfrac{1}{\sigma_s^2}\\
b_{s_0} &= 0\\
c_{s_0} &= 0
\end{aligned} \tag{2.43}
$$

with the same approximation ($\frac{\sin^2\frac{\phi}{2}}{\sigma_s^2} \approx 0$ so $a_x \approx \frac{\cos^2\frac{\phi}{2}}{\sigma_x^2}$) we integrate about x and the result is the next

$$
\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\int\int\sqrt{\frac{\pi\sigma_x^2}{\cos^2\frac{\phi}{2}}}e^{-a_1}e^{-(a_ss^2+b_ss+c_s)}e^{-(a_{s_0}s_0^2+b_{s_0}s_0+c_{s_0})}e^{\frac{-(d_1^2+d_2^2)}{2\sigma_x^2}}dsds_0 \tag{2.44}
$$

where

$a_1 = \frac{b_x^2}{4a_x} = (\frac{(d_1+d_2)\cos(\frac{\phi}{2})}{\sigma_x^2} - \frac{2s_0 \sin^2(\frac{\phi}{2})}{\sigma_s^2})^2 \frac{\sigma_x^2}{4\cos^2(\frac{\phi}{2})}$ we know that $\sin^2(\frac{\phi}{2})\sigma_x^2 \approx 0$

That results

$$a_1 = \frac{b_x^2}{4a_x} = (\frac{(d_1+d_2)\cos(\frac{\phi}{2})}{\sigma_x^2})^2 \frac{\sigma_x^2}{4\cos^2(\frac{\phi}{2})} \tag{2.45}$$

No making a little of algebra with (2.39) and with (2.40), we have that

$$\begin{aligned} d_1 + d_2 = & \frac{1}{K_{cr}} \sin(K_{cr}(s - s_0)) \sin(\frac{\phi}{2}) - \frac{1}{K_{cr}} \sin(K_{cr}(s + s_0)) \sin(\frac{\phi}{2}) \\ = & \frac{-2}{K_{cr}} \cos(K_{cr}s) \sin(K_{cr}s_0) \sin(\frac{\phi}{2}) \end{aligned} \tag{2.46}$$

Here we use the relations:

$$\begin{aligned} \sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a) \\ \cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \end{aligned}$$

So a_1 becomes,

$$\begin{aligned} a_1 = & (\frac{-2}{K_{cr}} \cos(K_{cr}s) \sin(K_{cr}s_0) \sin(\frac{\phi}{2}) \frac{\cos(\frac{\phi}{2})}{\sigma_x^2})^2 \frac{\sigma_x^2}{4\cos^2(\frac{\phi}{2})} \\ = & \frac{\cos^2(K_{cr}s) \sin^2(K_{cr}s_0) \sin^2(\frac{\phi}{2})}{K_{cr}^2 \sigma_x^2} \end{aligned} \tag{2.47}$$

In the same way for b_s

$$\begin{aligned} d_2 - d_1 = & -\frac{1}{K_{cr}} \sin(K_{cr}(s + s_0)) \sin(\frac{\phi}{2}) - \frac{1}{K_{cr}} \sin(K_{cr}(s - s_0)) \sin(\frac{\phi}{2}) \\ = & \frac{-2}{K_{cr}} \sin(K_{cr}s) \cos(K_{cr}s_0) \sin(\frac{\phi}{2}) \end{aligned} \tag{2.48}$$

and we have

$$d_1^2 = \frac{\sin^2(\frac{\phi}{2})(1-\cos(2K_{cr}(s-s_0)))}{2K_{cr}^2} \tag{2.49}$$

and

$$d_2^2 = \frac{\sin^2(\frac{\phi}{2})(1-\cos(2K_{cr}(s+s_0)))}{2K_{cr}^2} \tag{2.50}$$

we replace the values of a_s,b_s,c_s,a_{s_0},b_{s_0}, d_1^2, d_2^2 and c_{s_0} in (2.44) results

$$\begin{aligned}\mathcal{L} = \ & \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\sqrt{\frac{\pi\sigma_x^2}{\cos^2\frac{\phi}{2}}}\int\int exp(-s^2(\frac{\sin^2(\frac{\phi}{2})}{\sigma_x^2}+\frac{\cos^2(\frac{\phi}{2})}{\sigma_s^2}) \\ & +s(\frac{2\sin(K_{cr}s)\cos(K_{cr}s_0)\sin^2(\frac{\phi}{2})}{K_{cr}\sigma_x^2})-(\frac{s_0^2}{\sigma_s^2})+\frac{\cos^2(K_{cr}s)\sin^2(K_{cr}s_0)\sin^2(\frac{\phi}{2})}{K_{cr}^2\sigma_x^2} \\ & -\frac{\sin^2(\frac{\phi}{2})(1-\cos(2K_{cr}(s-s_0)))}{4K_{cr}^2\sigma_x^2}-\frac{\sin^2(\frac{\phi}{2})(1-\cos(2K_{cr}(s+s_0)))}{4K_{cr}^2\sigma_x^2})dsds_0\end{aligned} \tag{2.51}$$

and we define $N_1 = N_2 = N$

$$\mathcal{L}_{\prime} = \frac{N^2fN_b}{4\pi\sigma_x\sigma_y} \tag{2.52}$$

Thats results

$$\begin{aligned}\mathcal{L} = \ & \mathcal{L}_{\prime}\frac{\cos(\frac{\phi}{2})}{\pi\sigma_s^2}\int\int exp(-\frac{s^2\cos^2(\frac{\phi}{2})}{\sigma_s^2}-\frac{s_0^2}{\sigma_s^2}-\frac{\sin^2(\frac{\phi}{2})}{4K_{cr}^2\sigma_x^2}\{4K_{cr}^2s^2-8sK_{cr}\sin(K_{cr}s)\cos(K_{cr}s_0) \\ & -4\cos^2(K_{cr}s)\sin^2(K_{cr}s_0)+2-\cos(2K_{cr}(s-s_0))-\cos(2K_{cr}(s+s_0))\}dsds_0\end{aligned} \tag{2.53}$$

2.4.2 Crosing Angle with a CC for just one Beam.

For the last we calculate the Luminosity for just one beam is crabed, so we take that the beam 1 is crabbed.

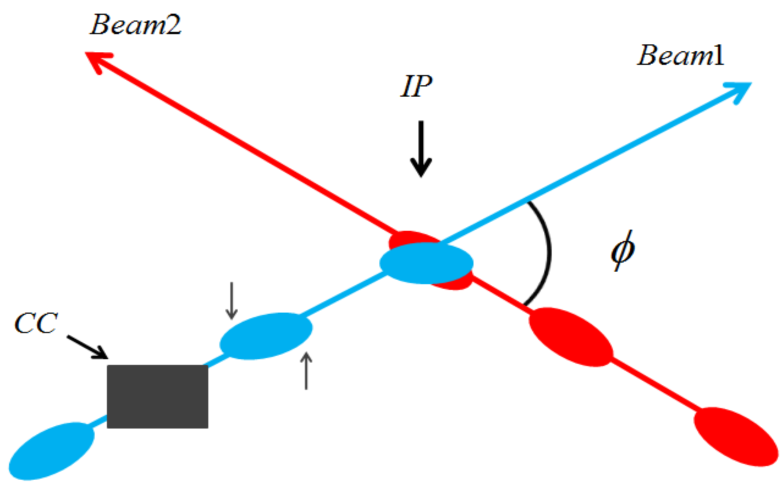

Figure 2.6: Scheme of the collision on two bunch with crossing angles and with one beam crabbed

For this case the coordiantes are

$$\begin{aligned} x_1 &= x\cos\left(\tfrac{\phi}{2}\right) - s\sin\left(\tfrac{\phi}{2}\right) + \tfrac{1}{K_{cr}}\sin(K_{cr}(s-s_0))\sin(\tfrac{\phi}{2}) \\ s_1 &= s\cos\left(\tfrac{\phi}{2}\right) + x\sin\left(\tfrac{\phi}{2}\right) \\ x_2 &= x\cos\left(\tfrac{\phi}{2}\right) + s\sin\left(\tfrac{\phi}{2}\right) \\ s_2 &= s\cos\left(\tfrac{\phi}{2}\right) - x\sin\left(\tfrac{\phi}{2}\right) \end{aligned} \tag{2.54}$$

The luminosity becomes:

$$\mathcal{L} = \frac{2\cos^2\left(\frac{\phi}{2}\right)N_1N_2fN_b}{\sqrt{2\pi}^6\sigma_x^2\sigma_y^2\sigma_s^2}\int\int\int\int e^{\frac{-(x_1)^2}{2\sigma_x^2}}e^{\frac{-(x_2)^2}{2\sigma_x^2}}e^{\frac{-(s_1-s_0)^2}{2\sigma_s^2}}e^{\frac{-(s_2+s_0)^2}{2\sigma_x^2}}e^{\frac{-y^2}{\sigma_y^2}}dxdydsds_0 \tag{2.55}$$

we call

$$d_1 = \frac{1}{K_{cr}}\sin(K_{cr}(s-s_0))\sin(\frac{\phi}{2}) \tag{2.56}$$

with

$$
\begin{array}{l}
x_1^2 = (x\cos(\frac{\phi}{2}) - s\sin(\frac{\phi}{2}) + d_1)^2 = x^2\cos^2(\frac{\phi}{2}) - 2xs\cos(\frac{\phi}{2})\sin(\frac{\phi}{2}) + s^2\sin^2(\frac{\phi}{2}) + d_1^2 \\
+2d_1(x\cos(\frac{\phi}{2}) - s\sin(\frac{\phi}{2})) \\
(s_1 - s_0)^2 = (s\cos(\frac{\phi}{2}) + x\sin(\frac{\phi}{2}) - s_0)^2 = s^2\cos^2(\frac{\phi}{2}) + 2sx\cos(\frac{\phi}{2})\sin(\frac{\phi}{2}) + x^2\sin^2(\frac{\phi}{2}) \\
-2s_0(s\cos(\frac{\phi}{2}) + x\sin(\frac{\phi}{2})) + s_0^2 \\
x_2^2 = (x\cos(\frac{\phi}{2}) + s\sin(\frac{\phi}{2}))^2 = x^2\cos^2(\frac{\phi}{2}) + 2sx\cos(\frac{\phi}{2})\sin(\frac{\phi}{2}) + s^2\sin^2(\frac{\phi}{2}) \\
(s_2 + s_0)^2 = (s\cos(\frac{\phi}{2}) - x\sin(\frac{\phi}{2}) + s_0)^2 = s^2\cos^2(\frac{\phi}{2}) - 2sx\cos(\frac{\phi}{2})\sin(\frac{\phi}{2}) + x^2\sin^2(\frac{\phi}{2}) \\
+2s_0(s\cos(\frac{\phi}{2}) - x\sin(\frac{\phi}{2})) + s_0^2
\end{array}
\tag{2.57}
$$

And working in the same way we have.

So now we write the equation (2.56) in order to use the formula (2.22), we get the same equation of (2.23)

$$
\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\int\int\int e^{-(a_xx^2+b_xx+c_x)}e^{-(a_ss^2+b_ss+c_s)}e^{-(a_{s_0}s_0^2+b_{s_0}s_0+c_{s_0})}e^{\frac{-d_1^2}{2\sigma_x^2}}dxdsds_0 \tag{2.58}
$$

with

$$
\begin{array}{l}
a_x = \frac{\cos^2\frac{\phi}{2}}{\sigma_x^2} + \frac{\sin^2\frac{\phi}{2}}{\sigma_s^2} \\
b_x = \frac{d_1\cos(\frac{\phi}{2})}{\sigma_x^2} - \frac{2s_0\sin\frac{\phi}{2}}{\sigma_s^2} \\
c_x = 0 \\
a_s = \frac{\sin^2\frac{\phi}{2}}{\sigma_x^2} + \frac{\cos^2\frac{\phi}{2}}{\sigma_s^2} \\
b_s = \frac{-d_1\sin(\frac{\phi}{2})}{\sigma_x^2} \\
c_s = 0 \\
a_{s_0} = \frac{1}{\sigma_s^2} \\
b_{s_0} = 0 \\
c_{s_0} = 0
\end{array}
\tag{2.59}
$$

with the same approximation ($\frac{\sin^2\frac{\phi}{2}}{\sigma_s^2} \approx 0$ so $a_x \approx \frac{\cos^2\frac{\phi}{2}}{\sigma_x^2}$) we integrate about x and the result is the next

$$\mathcal{L} = \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\int\int\sqrt{\frac{\pi\sigma_x^2}{\cos^2\frac{\phi}{2}}}e^{-a_1}e^{-(a_s s^2+b_s s+c_s)}e^{-(a_{s_0}s_0^2+b_{s_0}s_0+c_{s_0})}e^{\frac{-d_1^2}{2\sigma_x^2}}dsds_0 \tag{2.60}$$

where

$a_1 = \frac{b_x^2}{4a_x} = (\frac{d_1\cos(\frac{\phi}{2})}{\sigma_x^2} - \frac{2s_0\sin^2(\frac{\phi}{2})}{\sigma_s^2})^2\frac{\sigma_x^2}{4\cos^2(\frac{\phi}{2})}$ we know that $\sin^2(\frac{\phi}{2})\sigma_x^2 \approx 0$

so that results

$$a_1 = \frac{b_x^2}{4a_x} = (\frac{d_1\cos(\frac{\phi}{2})}{\sigma_x^2})^2\frac{\sigma_x^2}{4\cos^2(\frac{\phi}{2})} \tag{2.61}$$

So we replace with the equation 2.56, we have that

$$a_1 = \frac{\sin^2(K_{cr}(s-s_0))\sin^2(\frac{\phi}{2})\cos^2(\frac{\phi}{2})}{K_{cr}^2\sigma_x^4}\frac{\sigma_x^2}{4\cos^2(\frac{\phi}{2})} = \frac{\sin^2(K_{cr}(s-s_0))\sin^2(\frac{\phi}{2})}{4K_{cr}^2\sigma_x^2} \tag{2.62}$$

and we have that

$$d_1^2 = \frac{\sin^2(K_{cr}(s-s_0))\sin^2(\frac{\phi}{2})}{K_{cr}} \tag{2.63}$$

So we replace the values of a_s,b_s,c_s,a_{s_0},b_{s_0}, d_1^2 and c_{s_0} on (2.44) results.

$$\begin{aligned}\mathcal{L} = \quad & \frac{2\cos^2(\frac{\phi}{2})N_1N_2fN_b}{\sqrt{2\pi}^4\sigma_x^2\sigma_s^2}\frac{1}{2\sqrt{\pi}\sigma_y}\sqrt{\frac{\pi\sigma_x^2}{\cos^2\frac{\phi}{2}}}\int\int exp(-s^2(\frac{\sin^2(\frac{\phi}{2})}{\sigma_x^2}+\frac{\cos^2(\frac{\phi}{2})}{\sigma_s^2}) \\ & +s(\frac{\sin(K_{cr}(s-s_0))\sin^2(\frac{\phi}{2})}{K_{cr}\sigma_x^2}) - (\frac{s_0^2}{\sigma_s^2}) + \frac{\sin^2(K_{cr}(s-s_0))\sin^2(\frac{\phi}{2})}{4K_{cr}^2\sigma_x^2} - \\ & \frac{\sin^2(K_{cr}(s-s_0))\sin^2(\frac{\phi}{2})}{2K_{cr}^2\sigma_x^2})dsds_0\end{aligned} \tag{2.64}$$

And we define for $N_1 = N_2 = N$

$$\mathcal{L}\prime = \frac{N^2 f N_b}{4\pi\sigma_x\sigma_y} \tag{2.65}$$

Thats results

$$\begin{aligned}
\mathcal{L} = \quad & \mathcal{L}\prime\frac{\cos(\frac{\phi}{2})}{\pi\sigma_s^2}\int\int exp(-\frac{s^2\cos^2(\frac{\phi}{2})}{\sigma_s^2} - \frac{s_0^2}{\sigma_s^2} - \frac{\sin^2(\frac{\phi}{2})}{4K_{cr}^2\sigma_x^2}(4K_{cr}^2 s^2 - 8sK_{cr}\sin(K_{cr}(s-s_0)) \\
& + \sin^2(K_{cr}(s-s_0))))dsds_0 \\
& = \mathcal{L}\prime\frac{\cos^2(\frac{\phi}{2})}{\pi\sigma_s^2}\int\int exp(-\frac{s^2\cos^2(\frac{\phi}{2})}{\sigma_s^2} - \frac{s_0^2}{\sigma_s^2} - \frac{\sin^2(\frac{\phi}{2})}{4K_{cr}^2\sigma_x^2}(-2K_{cr}s \\
& + \sin(K_{cr}(s-s_0)))^2)dsds_0
\end{aligned} \tag{2.66}$$

The Luminosity calculations are in agreement with the previous results [5] and [6].

Bibliography

[1] Alessandro Gallo, RF Systems I, II, Lectures CAS, 2008.

[2] S.Y.Lee, Accelerator Physics, World Scientific Publishing Co, Pte. Ltd,Second edition.

[3] F. Zimmermann and U. Dorda, in Proceedings of LHCLUMI- 05 Workshop, Arcidosso, Italy, 2005 (Report No. CARE-Conf-06-014-HHH).

[4] D.A. Edwards and M.J. Syphers, An Introduction to the Physics of High Energy Accelerators, Wiley Inter-Science,1993. Pages 8-10,74-78,91-93.

[5] Yi-Peng Sun, Phys. Rev. ST Accel. Beams 12, 101002 (2009).

[6] Werner Herr, Concept of Luminosity, Lectures CAS, 2009.